Formalising the initiality conjecture in Coq

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Göteborg–Stockholm Joint Type Theory Seminar
Chalmers / Göteborg University, 12 December 2018
Formalising the initiality conjecture in Coq and Agda

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Initiability for dependent type theories

By *initiability* for a type theory, mean a statement like:

**Template**

The syntactic category of dependent type theory with $\Sigma$, $\Pi$, and Id-types forms the initial contextual category with $\Sigma$, $\Pi$, and Id-structure.
By **initiality** for a type theory, mean a statement like:

**Template**

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- Justifies categorical-algebraic definition of “models of DTT with XYZ-types” as “contextual cats with XYZ-structure”
- Packages the bureaucracy of interpreting syntax into such structures
- Should hold uniformly for all dependent type theories
- Variations: could state with CwA’s, CwF’s, C-systems, etc.; with various different presentations of the type theory; with 2-categorical initiality; …
Status

- **Thesis: Initiality is established**
  - Proven by Streicher (1991 book) for Calculus of Constructions; by Hofmann (1995 thesis) for DTT with $\Pi$, $\text{Id}$, $\mathbb{N}$.
  - Proof extends straightforwardly + robustly to other type theories.
  - (NB: Other presentations in literature (that I’m aware of) use techniques specific to their particular type theories, don’t extend so robustly; or handwave many details.)
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  - Extension not really straightforward at all!
  - What type theories is it even supposed to hold for? It fails for some!
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▶ **Synthesis: Initiality is heuristically well-understood**
  - “Experts” do understand what kinds of type theories it holds for, and how to extend Streicher’s proof.
  - But: this understanding not clearly articulated anywhere, rigorously or even heuristically.
  - Extension of proof mostly straightforward — minor tweaks needed, no substantial new ideas — but carefully making sure of this involves checking a lot of details.
Solution proposals

<table>
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Solution proposals

Long-term solution

Define general class of dependent type theories; state and prove initiality for these.

- Precisely: define rigorously a general class of dependent type theories…
- …yielding, as instances, as many of the specific theories of interest as possible…
- …modulo minor differences in presentation, as minor as possible.
- Define corresponding categorical-algebraic structures for these…
- …yielding the established definitions, as closely as possible…
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Short-term solution

Just damn well prove it for some more of the specific type theories of interest!
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Proofs of initiality are like buses

Goal

Prove initiality carefully, for some specific type theory, ideally approaching “book HoTT”.

Major recent assault on this: Initiality Project.

Collaborative many-author project; led by Shulman; aiming for book HoTT; written, not formalised.


Small type theory at first: \( \Pi \)-types, a dependent family of base types.

Key design criterion: robust extensibility. Avoid doing anything that wouldn't extend to arbitrary constructors/rules.

“We can have this done within a week.” — PLL, 19 October 2018

Developments begun 22 October; Brunerie-de Boer initiality attained 19 November!
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Over to Guillaume!
[Here PLL hands over to Guillaume, who presents the Brunerie–de Boer Agda formalisation before handing back to PLL with the rest of these slides.]
Lumsdaine–Mörtberg formalisation details
Background


Attempt foundered due to combination of several design choices making life hard:

- use of named variables
- use of setoids for the target models
- started with slightly overambitious type theory
- ...

Very useful experience to build on — both the good and the bad aspects…
Design choices

This time round:

▶ Proof assistant: Coq; specifically, over UniMath. (Mainly: for a well-developed category theory library that both authors were familiar with.)

▶ Models: Categories with attributes, not assuming objects form a set (so, CwA a 2-category); for 1-categorical initiality, contextual categories as CwA’s plus contextuality axiom (implying objects a set).

▶ Variables in raw syntax: using de Bruijn indices. Raw syntax: well-scoped. These enable:

▶ All inductions purely structural, over either raw syntax or derivations. No size measures, auxiliary well-founded relations, etc.

▶ Context and context-equality judgements subsidiary, not primitive, don’t appear in derivations. Substitution admissible, not a primitive rule. These enable:

▶ Interpretation function (partial + totality): into arbitrary CwA’s. No use of equality on objects/contexts needed.
Experience

Good news

1. Interpretation function (partial + total): went very smoothly.
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Specifically, interaction of 2 issues:

1. Syntactic CwA a dependently-typed structure (maps depend on objects), where the objects are quotiented (contexts, up to judgemental equality)
2. In UniMath’s quotients, the dependent eliminator doesn’t compute judgementally.

Together: ends up about as painful as using setoids.
Status

1. “Pre-quotient” parts: almost all done, remaining part expected quick. Contains all the mathematically interesting bits.
   ▶ Done: definition of type theory, and suitable structured CwA’s; admissibility of substitution, presuppositions; structural operations of the syntactic CwA, and interpretation function into CwA’s/contextual categories, i.e. pre-quotient aspects of existence part of initiality.
   ▶ Remaining to do: functoriality of interpretation under CwA maps, amounting to uniqueness part of initiality.

Around 4,000 lines of code (not including libraries).

2. “Post-quotient” parts, i.e. assembling the pre-quotient parts into the syntactic CwA and functions thereon: some parts done, much remaining. Mathematically fairly uninteresting, but difficult and slow.
   ▶ Done: syntactic category; part of CwA structure thereon; most of underlying functor of the interpretation map.
   ▶ Remaining to do: rest of CwA structure, and logical structure thereon; interpretation as a map of CwA’s with structure; uniqueness of the interpretation map.

Arround 1,000 lines of code, so far!
Dinner: Restaurant Zozaki, Stora Nygatan 3, 18:00!