Higer Categories from Type Theories
(bis)

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Setting

(Martin-Löf) Dependent Type Theory: highly expressive constructive theory, potential foundation for maths.

Central concepts: *types*, and *terms* of *types*.

$$\vdash \mathbb{N} \ \mathsf{type} \qquad \vdash 0 : \mathbb{N}$$

Both can be *dependent* on (typed) variables:

$$n : \mathbb{N} \ \vdash \ \mathbb{R}^n \ \ \mathsf{type}$$

"For each n in $\mathbb{N}$, $\mathbb{R}^n$ is a type," or "$\mathbb{R}^n$ is a type, dependent on $n : \mathbb{N}$."

$$n : \mathbb{N} \vdash \mathbf{0}_n : \mathbb{R}^n$$

"For each n in $\mathbb{N}$, $\mathbf{0}_n$ is an element of $\mathbb{R}^n$."

$$\vdash \mathbf{0} : \prod_n \mathbb{R}^n$$

Identity types

Logic: via Curry-Howard, predicates as dependent types.

Predicate of equality, identity:

$$x, y : A \vdash \mathrm{Id}_A(x, y) \ \mathsf{type}$$

Can derive e.g. "transitivity of equality",

$$x, y, z : A,\ u : \mathrm{Id}(x, y),\ v : \mathrm{Id}(y, z) \ \vdash\ c(u, v) : \mathrm{Id}(x, z)$$

"functions respect equality",

$$\frac{x : A \vdash f(x) : B}{x, y : A,\ u : \mathrm{Id}(x, y) \vdash f^{*}u : \mathrm{Id}(f(x), f(y))}$$

and much more…

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Question: How much more?

Higher Categories from Types

Two subtleties:

- Identity types may be non-trivial types: not all identity proofs equal.
- Identity types have higher identity types in turn:

$$x, y : A,\ u, v : \mathrm{Id}_A(x, y) \vdash \mathrm{Id}_{\mathrm{Id}_A(x,y)}(u, v).$$

Compositions of propositional equalities over a single type:

Theorem (Garner-van den Berg, PLL)

For any DTT $\boldsymbol{T}$ with Id*-types, and any type A of $\boldsymbol{T}$, A and its tower of identity types form an internal ω-groupoid in $\boldsymbol{T}$.*

(All ω-categories: weak, globular operadic à la Batanin/Leinster.)

Higher Categories from Type Theories

Across all types of a theory?

Definition

Given $\mathbf{T}$, define globular set $\mathcal{C}\ell_\omega^{\mathrm{ty}}(\mathbf{T})$ by:

- 0-cells: closed types $\vdash A\ \mathsf{type}$;
- 1-cells: terms $x : A \vdash f(x) : B$;
- 2-cells: terms $x : A \vdash \alpha(x) : \mathrm{Id}_B(f(x), g(x))$;
- etc…

Similarly, $\mathcal{C}\ell_\omega(\mathbf{T})$: same but with contexts, not just types, as 0-cells.

1-skeleton of this underlies the classifying category $\mathcal{C}\ell(\mathbf{T})$.

Theorem (PLL)

For any $\boldsymbol{T}$ with Id-types and extensional Π-types, $\mathcal{C}\ell_\omega(\boldsymbol{T})$ underlies an ω-category, groupoidal in dimensions ≥ 2.

Take-home points

Three formal devices allow one to isolate the proof-theoretic content. None new, but all could be better-known:

1. Type theories form a category.
2. Contexts are just like types.
3. Conservativity is a lifting property.

Categories of Type Theories

Definition

A *type system* Φ is, informally, a collection of constructors and rules, e.g. "Id-types and extensional Π-types".

Formally: an essentially algebraic theory extending the theory of contextual categories, with the same sorts.

Given such Φ, write $\mathbf{DTT}_\Phi$ for the category of type theories given by the constructors of Φ plus possibly further *algebraic* axioms, and translations between such theories preserving the constructors of Φ.

As models of an essentially algebraic theory, each $\mathbf{DTT}_\Phi$ is locally presentable; in particular, co-complete.

For extension of type systems $\Phi \longrightarrow \Xi$, have evident adjunction

$$\mathbf{DTT}_\Phi \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{DTT}_\Xi \;.$$

From contexts to types

For many nice type systems Φ, all the constructors/rules lift from types to contexts, so have a functor

$$(-)^{\mathrm{cxt}} : \mathbf{DTT}_\Phi \longrightarrow \mathbf{DTT}_\Phi$$

where $\mathbf{T}^{\mathrm{cxt}}$ is the theory whose types are the contexts of $\mathbf{T}$.

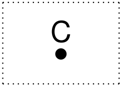
Then $\mathcal{C}\ell_\omega(\mathbf{T}) \cong \mathcal{C}\ell_\omega^{\mathrm{ty}}(\mathbf{T}^{\mathrm{cxt}})$, so to construct algebraic structure on $\mathcal{C}\ell_\omega$, it's enough to construct it on $\mathcal{C}\ell_\omega^{\mathrm{ty}}$.

(Typically, $(-)^{\mathrm{cxt}}$ is nearly but not quite a monad: its multiplication "map" fails to preserve constructors on the nose.)

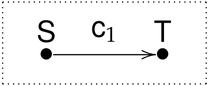
The type-theoretic globes

Fix Φ. Define theories $\mathbf{g}_n$ over Φ by axioms:

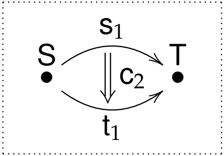
$\mathbf{g}_0$: $\vdash \mathsf{C}\ \mathsf{type}$



$\mathbf{g}_1$:
$$\vdash \mathsf{S}, \mathsf{T}\ \mathsf{type}$$
$$x : \mathsf{S} \vdash \mathsf{c}_1(x) : \mathsf{T}$$



$\mathbf{g}_2$:
$$\vdash \mathsf{S}, \mathsf{T}\ \mathsf{type}$$
$$x : \mathsf{S} \vdash \mathsf{s}_1(x), \mathsf{t}_1(x) : \mathsf{T}$$
$$x : \mathsf{S} \vdash \mathsf{c}_2(x) : \mathrm{Id}_{\mathsf{T}}(\mathsf{s}_1(x), \mathsf{t}_1(x))$$



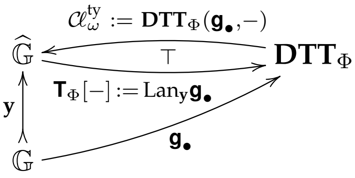
etc.

These form a *coglobular theory*: $\mathbf{g}_\bullet : \mathbb{G} \longrightarrow \mathbf{DTT}_\Phi.$

In fact, $\mathbf{g}_\bullet$ represents $\mathcal{C}\ell_\omega^{\mathrm{ty}}$: $\mathbf{DTT}_\Phi(\mathbf{g}_n, \mathbf{T}) \cong \mathcal{C}\ell_\omega^{\mathrm{ty}}(\mathbf{T})_n.$

Representability

Induced Kan situation:



The left Kan extension $\mathbf{T}_\Phi[-] \ := \ \mathrm{Lan}_{\mathbf{y}} \mathbf{g}_\bullet$ gives *logical realisations* of globular sets as theories over Φ.

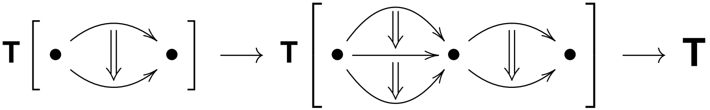
To put a natural ω-category structure on $\mathcal{C}\ell_\omega^{\mathrm{ty}}$, equivalent to put a co-ω-category structure on $\mathbf{g}_\bullet$.

So: want to find *contractible globular operad* P acting on $\mathbf{g}_\bullet$; that is, with a map $P \longrightarrow \mathrm{End}(\mathbf{g}_\bullet)$, implementing elements of P as *composition co-operations* on $\mathbf{g}_\bullet$.

Composition co-operations

What is a composition co-operation on $\mathbf{g}_\bullet$ for a pasting diagram π, and how does it induce a composition operation for π on $\mathcal{C}\ell_\omega^{\mathrm{ty}}$?

Might first expect: a map $\mathbf{g}_n \longrightarrow \mathbf{T}[\hat{\pi}]$, from the n-globe into the realisation of π, inducing operation by precomposition.



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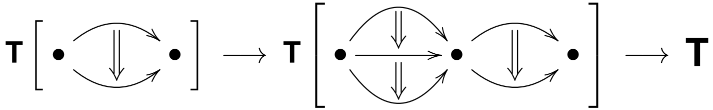
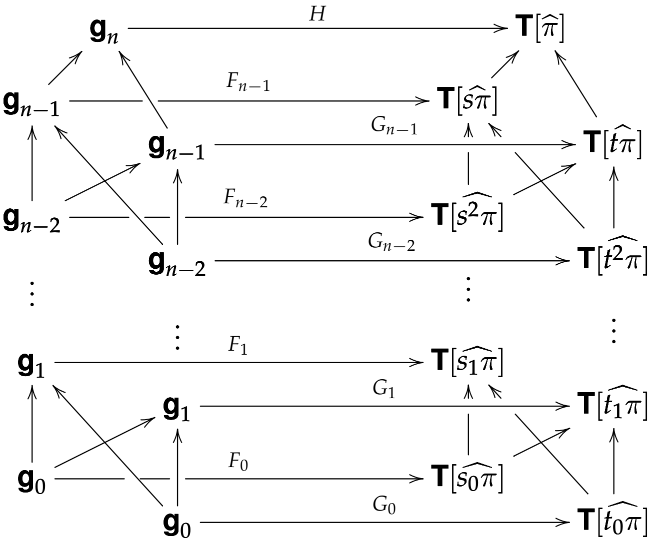


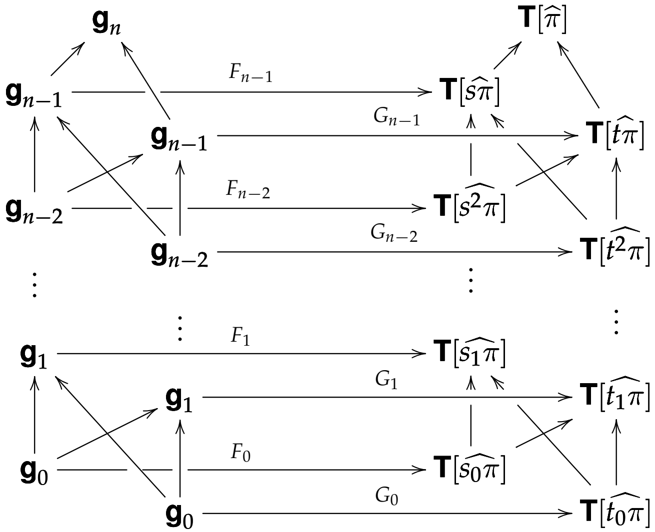
Roughly right... but need also to specify how it acts in lower dimensions.

Composition co-operations



Contractibility for co-operations

Contractibility in End($\mathbf{g}_\bullet$) means always being able to fill apex:



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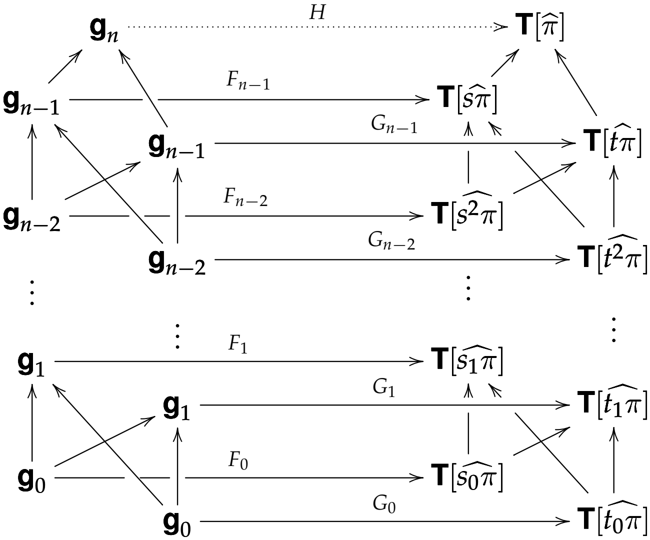


A contractible sub-operad

Simplifying the picture, need to fill certain ‘triangles’:

$$\begin{array}{ccc} \mathbf{g}_n & \dashrightarrow & \mathbf{T}[\hat{\pi}] \\ \uparrow & & \uparrow \\ \partial \mathbf{g}_n & \longrightarrow & \mathbf{T}[\partial \hat{\pi}] \end{array}$$

“Given co-operations for composing the boundary of π, need to complete to a co-operation for π.”

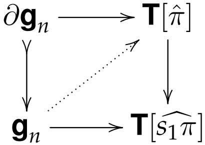
Let $P \subseteq \mathrm{End}(\mathbf{g}_\bullet)$ be the sub-operad of co-operations which ‘do the obvious thing’ on dimensions ≤ 1.

Goal

The sub-operad P is contractible.

Contractibility to contractibility

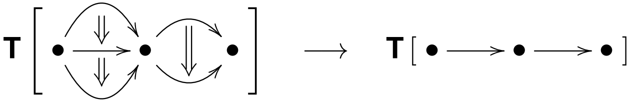
For co-operations in P, the triangle problem above fits into a square-filling problem:



(Here $s_1\pi$ denotes the 1-dimensional source/target of π; the square commutes by definition of P.)

So contractibility of P reduces to "contractibility" — a right lifting property — for maps of theories

$$\mathbf{T}[\hat{\pi}] \quad \longrightarrow \quad \mathbf{T}[\widehat{s_1\pi}].$$



Contractibility as conservativity

Concretely, the desired right lifting property

$$\begin{array}{ccc} \partial\mathbf{g}_n & \longrightarrow & \mathbf{T} \\ \downarrow & \nearrow & \downarrow \\ \mathbf{g}_n & \longrightarrow & \mathbf{S} \end{array}$$

is a *conservativity* principle: given some type in $\mathbf{T}$, inhabited in the extension $\mathbf{S}$, want to lift this inhabitant to $\mathbf{T}$.

So, reduced to proof-theoretic crux:

Lemma

If the maps $\boldsymbol{T}[\hat{\pi}] \longrightarrow \boldsymbol{T}[\widehat{s_1\pi}]$ are conservative, then P is a contractible sub-operad of $\mathrm{End}(\boldsymbol{g}_\bullet)$*, and hence $\mathcal{C}\ell_\omega$ carries a natural ω-category structure.*

Main theorem

Corollary

If Φ *contains* Id*-types and extensional* Π*-types, then these maps are conservative, so* $\mathcal{C}\ell_\omega$ *is naturally an* ω*-category, as desired.*

Conjecture

If Φ *consists of just* Id*-types, these maps are again conservative. Hence, for any* Ξ *containing at least* Id*-types,* $\mathcal{C}\ell_\omega$ *carries the desired* ω*-category structure, via the adjunction* $\mathbf{DTT}_{\mathrm{Id}} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{DTT}_{\Xi}$ *.*

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Take-home points, again:

1. Type theories form a category.
2. Contexts are just like types.
3. Conservativity is a lifting property.

Thank you!

These slides, plus thesis (containing details omitted here), available from:

http://www.mathstat.dal.ca/~p.l.lumsdaine