Conservativity Principles:

a Homotopy-Theoretic Approach

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Outline

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   - A model structure on DTT's?

Conservativity, classically

Definition

An extension $\mathcal{T} \subseteq \mathcal{S}$ of (propositional, predicate) theories is conservative if:

- for every proposition A of $\mathcal{T}$ that is a theorem of $\mathcal{S}$,
- A is already a theorem of $\mathcal{T}$.

Example (Extension by definitions)

$\mathcal{T}$ any theory, τ any term of $\mathcal{T}$. Let $\mathcal{T}[t := \tau]$ be $\mathcal{T}$ plus a new symbol t and new axiom $t = \tau$. Then $\mathcal{T}[t := \tau]$ is conservative over $\mathcal{T}$.

Conservativity, categorically

Definition

A morphism of theories $F : \mathcal{T} \to \mathcal{S}$ is conservative if

- for every proposition A of $\mathcal{T}$ s.t. $F(A)$ is a theorem of $\mathcal{S}$,
- A is a theorem of $\mathcal{T}$.

Example (Extension by definitions)

Fact. The inclusion $\mathcal{T} \hookrightarrow \mathcal{T}[t := \tau]$ is conservative.
Proof. It has a retraction $\mathcal{T}[t := \tau] \to \mathcal{T}$.
Fact. This retraction $\mathcal{T}[t := \tau] \to \mathcal{T}$ is itself conservative.
Fact. Indeed, $\mathcal{T}[t := \tau] \cong \mathcal{T}$.

Conservativity in dependent type theories

In DTT: various possbile generalisations of conservativity. Not just *existence* of proofs, but *equality of proofs*?

Definition (Hofmann, [Hof97])

A morphism of theories $F\colon \mathcal{T} \to \mathcal{S}$ is (strongly conservative?) if whenever $\Gamma \ \vdash_{\mathcal{T}} \ A$ type and $F(\Gamma) \ \vdash_{\mathcal{S}} \ a : F(A)$, there is some term $\overline{a}$ with $\Gamma \ \vdash_{\mathcal{T}} \ \overline{a} : A$ and $F(\Gamma) \ \vdash_{\mathcal{S}} \ F(\overline{a}) = a : F(A)$.

Can also consider (weakly conservative?), with second clause of conclusion omitted; also, similar conservativity clauses with *types* as well as *terms*.

Can also weaken second clause of conclusion to *propositional* equality.

Extensions by definitions in DTT

New term *definitionally* equal to old, or just *propositionally*?

Example (Extension by “definitional definitions”)

Just as before — $\mathcal{T}[\vec{x} : \Gamma \vdash a(\vec{x}) := \alpha(\vec{x}) : A(\vec{x})] \cong \mathcal{T}$.

Example (Extension by “propositional definitions”)

$\mathcal{T}[\vec{x} : \Gamma \vdash a(\vec{x}) :\simeq \alpha(\vec{x}) : A(\vec{x})]$ — extension of $\mathcal{T}$ by terms

$$\Gamma \vdash a(\vec{x}) : A(\vec{x}) \qquad \Gamma \vdash l(\vec{x}) : \mathrm{Id}_A(a(\vec{x}), \alpha(\vec{x})).$$

Have inclusion, retraction $\mathcal{T} \hookrightarrow \mathcal{T}[a :\simeq \alpha] \twoheadrightarrow \mathcal{T}$ as before. Hence, inclusion is *weakly conservative*.

Retraction? When Γ empty, *strongly conservative* by Id-ELIM, since adjoining closed terms is just declaring variables.

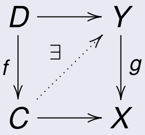
When Γ non-empty…?? Surpisingly hard!

Weak lifting properties

A tool from homotopy theory:

Definition

$\mathcal{C}$ a category, f, g maps. Say $f \pitchfork g$ if every square from f to g has a filler:



aka "f has (weak) left lifting property against g", "f (weakly) left orthogonal to g", etc.

Typically, *cofibrations* have left lifting properties, *fibrations* have right lifting properties.

Example: topological spaces

In **Top**, boundary inclusions of discs:

$$i_n \colon S^{n-1} \hookrightarrow D^n \qquad n \geq 0.$$

Definition

A map $p \colon Y \to X$ is a (Quillen) trivial fibration (aka weakly contractible) if it is right orthogonal to each i_n:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & Y \\ {\scriptstyle i_n}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle p} \\ D^n & \longrightarrow & X \end{array}$$

Implies: p a weak homotopy equivalence.

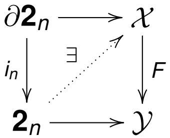
Example: *n*-categories

In *n*-**Cat**, boundary inclusions of cells:

$$i_n \colon \partial \mathbf{2}_n \hookrightarrow \mathbf{2}_n \qquad n \geq 0.$$

Definition

A map $F \colon Y \to X$ is a (Joyal/Lack/etc.) trivial fibration (aka contractible) if it is right orthogonal to each i_n:



In **Cat**, precisely: F full, faithful, surjective.

Dependent Type Theories

Definition

DTT: category of dependent type theories *(all algebraic extensions of some fixed set of constructors)* and interpretations.

Basic judgements: $\Gamma \vdash A$ type $\quad \Gamma \vdash a : A$.

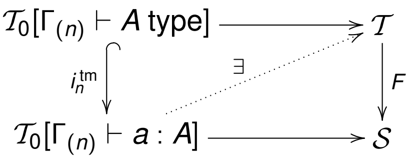
Judgments have boundaries too!
and again these are (familially) representable:

$$\begin{array}{rcl} i_n^{\mathrm{ty}} : \mathcal{T}_0[\Gamma_{(n)}] & \hookrightarrow & \mathcal{T}_0[\Gamma_{(n)} \vdash A \text{ type}] \\ i_n^{\mathrm{tm}} : \mathcal{T}_0[\Gamma_{(n)} \vdash A \text{ type}] & \hookrightarrow & \mathcal{T}_0[\Gamma_{(n)} \vdash a : A] \end{array} \qquad n \geq 0$$

Contractible maps of theories

Definition

$F \colon \mathcal{T} \to \mathcal{S}$ is term-contractible if it is right orthogonal to each basic term inclusion $i_n^{\mathrm{tm}} \colon \mathcal{T}_0[\Gamma_{(n)} \vdash A \text{ type}] \hookrightarrow \mathcal{T}_0[\Gamma_{(n)} \vdash a : A]$.
Similarly: type-contractible, contractible.



Flashback

$F \colon \mathcal{T} \to \mathcal{S}$ is (strongly conservative?) if whenever $\Gamma \vdash_{\mathcal{T}} A$ type and $F(\Gamma) \vdash_{\mathcal{S}} a : F(A)$, there is some term $\bar{a}$ with $\Gamma \vdash_{\mathcal{T}} \bar{a} : A$ and $F(\Gamma) \vdash_{\mathcal{S}} F(\bar{a}) = a : F(A)$.

Realisation

"Term-contractible" is exactly "strongly conservative"!

Now, fix constructors: Id-types, Π-types, and functional extensionality ("functions are equal if equal on values", [AMS07]; nothing to do with "extensionality principles" like reflection rule). (Or, set of constructors extending these.)

Lemma

For any "extension by propositional definition", the retraction

$$\mathcal{T}[a(\vec{x}) :\simeq \alpha(\vec{x})] \longrightarrow\!\!\!\!\!\rightarrow \mathcal{T}$$

is term-contractible.

Extensions by propositional definitions, revisited

Lemma

For any "extension by propositional definition", the retraction

$$\mathcal{T}[a(\vec{x}) :\simeq \alpha(\vec{x})] \twoheadrightarrow \mathcal{T}$$

is term-contractible.

Proof

Reduce to known closed case, via retract argument:

$$\begin{array}{ccc} \mathcal{T}[a(\vec{x}) :\simeq \alpha(\vec{x})] & \rightleftarrows & \mathcal{T}[f :\simeq \lambda \vec{x}.\, \alpha(\vec{x})] \\ \downarrow\!\!\downarrow & & \downarrow \\ \mathcal{T} & = & \mathcal{T} \end{array}$$

□

Classifying weak ω-categories

Above lemma is key to construction of higher categories from dependent type theories:

Theorem

If **DTT** *is any category of dependent theories with* Id*-types and satisfying the lemma above (e.g.* $\mathbf{DTT}_{\mathrm{Id},\Pi,\mathsf{fext}}$*), then there is a functor*

$$\mathbf{DTT} \xrightarrow{\mathbf{Cl}_\omega} \mathbf{wk}\text{-}\omega\text{-}\mathbf{Cat}$$

giving the classifying weak ω*-category of a theory* $\mathcal{T} \in \mathbf{DTT}$*.*

(Objects of $\mathbf{Cl}_\omega(\mathcal{T})$ are contexts; 1-cells are context morphisms; higher cells are constructed from terms of identity types.)

Model structures

The model structures on n-**Cat** (Joyal–Tierney, Lack, Lafont–Métayer–Worytkiewicz), and some others, can be uniformly constructed purely in terms of their generating cofibrations—the basic inclusions of boundaries into cells. (But proving they are model structures is hard in each case!)

Question

Does the same construction, applied to these "type-theoretic boundary inclusions" i_n^{tm}, i_n^{ty}, give a model structure on **DTT**?

From this point of view, above lemma shows that *pushouts of certain trivial cofibrations are again weak equivalences*!

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