Topology and Algebra

An Introduction to Duality

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Outline

The Idea of Duality

Slogan

Now, commutative algebra is like topology, only backwards. —John Baez

Represent spaces (and maps between them) as algebras (and maps *going the other way*):

Spaces	Algebras
$X \xrightarrow{f} Y$	$A \xleftarrow{\varphi} B$

This idea gives: algebraic geometry, non-commutative geometry/topology, K-theory, point-free topology, . . .

The Point of the Idea of Duality

Algebra:

- Unified framework
- Technically flexible
- Constructive/computable

Geometry/Topology:

- Intuition!
- Pictures!

Different categories of spaces correspond to different categories of algebras.

Example

Gel'fand Duality between compact Hausdorff spaces and commutative C^*-algebras.

C^*-Algebras

Definition

A C^*-algebra A is a Banach space $(A, \|-\|)$ together with a continuous, bilinear multiplication making A a ring, and an operation $(-)^* : A \to A$ such that:

- $(-)^*$ is anti-linear: $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \bar{\lambda} a^*$;
- $(-)^*$ is an anti-homomorphism: $1^* = 1$, $(ab)^* = b^* a^*$;
- $(-)^*$ is an involution: $a^{**} = a$; and
- the C^**-identity*: $\|a^* a\| = \|a\|^2$.

Example

- $\mathbb{C}$, with usual norm and ring structure, and conjugation as $(-)^*$.

The C^*-Algebra of Functions on a Compact Space

Definition

Briefly: A C^*-algebra A is a Banach algebra with an involutive anti-linear anti-endomorphism $(-)^*$ satisfying the C^*-identity.

Example

X a compact space. $C(X) := \{\text{continuous maps } f\colon X \to \mathbb{C}\}$.
Then $C(X)$, with the supremum norm $\| - \|_\infty$, and pointwise addition, multiplication and conjugation, is a commutative C^*-algebra.

Question

Is every commutative C^*-algebra of this form?

The Spectrum of a Commutative C^*-Algebra

Definition

A a commutative C^*-algebra.

$$\mathrm{Spec}(A) := \{C^*\text{-homomorphisms } \alpha\colon A \to \mathbb{C}\},$$

topologised as a subset of A^* with weak-* topology.

Proposition

Spec(A) is a compact Hausdorff space.

Question

How do these two constructions interact?

The Gel'fand Representation Theorem

Theorem (Gel'fand Representation Theorem)

- *X a compact Hausdorff space; then* $\mathrm{Spec}(C(X)) \cong X$.
- *A a commutative C^*-algebra; then* $C(\mathrm{Spec}(A)) \cong A$.

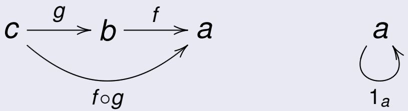
So, we have a correspondence between (homeomorphism classes of) compact Hasdorff spaces and (isomorphism classes of) commutative C^*-algebras.
(Also extends to a correspondence between *locally compact* spaces and *non-unital* C^*-algebras.)

More than this: it's an *equivalence of categories*, so preserves much important structure.

Categories

Definition

A category **C** consists of collections ob **C** (“objects”) and arr **C** (“arrows”) equipped with unital, associative composition.



(NB Objects, arrows can be anything, like elements of a group.)

Example

- **Set**: sets, functions
- **Top**: top. spaces, cts maps
- **Gp**: groups, homo'isms
- $\mathbf{Vect}_k$: k-vector spaces, linear maps

Functors

Definition

A functor (map of categories) $F \colon \mathbf{C} \to \mathbf{D}$ consists of maps from the objects and arrows of $\mathbf{C}$ to those of $\mathbf{D}$, preserving units and composition.

$$b \xrightarrow{f} a \qquad \longmapsto \qquad F(b) \xrightarrow{F(f)} F(a)$$

Example

- "underlying set" $U \colon \mathbf{Gp} \to \mathbf{Set}$
- "free group" $F \colon \mathbf{Set} \to \mathbf{Gp}$
- "homology groups" $H_\bullet \colon \mathbf{Top} \to \mathbf{AbGp}^{\mathbb{Z}}$

Contravariant functors

Definition

For any category **C**, the dual category $\mathbf{C}^{\text{op}}$ has the same objects and arrows, but with the source and target of each arrow swapped.

A contravariant functor fom **C** to **D** is a functor from $\mathbf{C}^{\text{op}}$ to **D**, or equivalently from **C** to $\mathbf{D}^{\text{op}}$.

$$b \xrightarrow{f} a \qquad \longmapsto \qquad F(b) \xleftarrow{F(f)} F(a)$$

Example

- "dual" $(-)^* : \mathbf{Vect}_k^{\text{op}} \to \mathbf{Vect}_k$
- "function algebra" $C(-) : \mathbf{CpctHaus} \to \mathbf{Comm}\text{-}C^*\text{-}\mathbf{Alg}^{\text{op}}$
- "spectrum" $\text{Spec} : \mathbf{Comm}\text{-}C^*\text{-}\mathbf{Alg}^{\text{op}} \to \mathbf{CpctHaus}$

Isomorphisms and Equivalences

Definition

An isomorphism $a \cong b$ in $\mathbf{C}$ is a pair of arrows $b \underset{g}{\overset{f}{\rightleftarrows}} a$ with $f \circ g = 1_a$ and $g \circ f = 1_b$.

"Isomorphism", not equality, is usually the appropriate notion of "same" for objects. For categories, a weaker notion still:

Definition

An equivalence of categories $\mathbf{C} \simeq \mathbf{D}$ is a pair of functors

$$\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$$

together with natural isomorphisms $GF(c) \cong c$, $FG(d) \cong d$ for objects c, d of $\mathbf{C}$, $\mathbf{D}$.

Gel’fand Duality

Theorem (Gel’fand Duality)

$$\mathbf{CpctHaus} \underset{\mathrm{Spec}}{\overset{C(-)}{\rightleftarrows}} \mathbf{Comm}\text{-}C^*\text{-}\mathbf{Alg}^{\mathrm{op}}$$

is an equivalence of categories.

These categories are the same: commutative C^*-algebras, and homormorphisms backwards between them, can be seen as an alternative presentation of compact Hausdorff spaces and continuous maps.

Dictionary Between Topology and Algebra

This equivalence allows us to translate many concepts from topology into the world of C^*-algebras:

Example

Loc. cpct. Hausd. space	Nonunital comm. C^*-algebra
Compact	Unital
Continuous proper map	C^*-homomorphism (backwards)
Homeomorphism	Isomorphism
Metrizable	Separable
Vector bundle	Projective module
Measure space	von Neumann algebra
...	...

Consequences of Gel'fand Duality

- Topological constructions: often easier in the language of algebra.
- Algebraic constructions: often more intuitive seen geometrically.
- Noncommutative Topology à la Connes. Look at *all* C^*-algebras. Our translation gives language to discuss these geometrically. Non-commutative ones model geometric objects not represented well by ordinary topological spaces: e.g. "space" of Penrose tilings, Heisenberg operator algebras.

Further Dualities

Various dualities treat other categories of spaces, to extend theories in other directions:

- Noncommutative Topology: C^* algebras; compact Hausdorff spaces.
- Algebraic Geometry: algebraic varieties; commutative rings.
- Noncommutative Differential Geometry: manifolds (smooth, analytic, ...); various classes of algebras.
- Point-free Topology: sober topological spaces; locales (= complete ($\wedge$-$\bigvee$)-distributive lattices).

In Conclusion. . .

Moral

Trading spaces for algebras, through the mirror of duality, gives us new perspectives and allows us to represent new classes of geometrical objects.