(Stasheff & Boardman, Vogt, May)

→ Last time, we defined a topological operad. Now, let's drop the topological, & take a look at what we get.

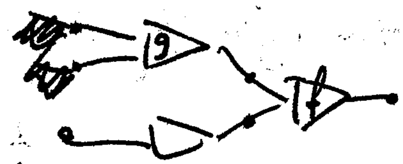
This is ~~aaaa~~ so far just a funny language for commonplace univ. algebra: —

to be precise, strongly regular theories up to presentation

An operad P is ~~aa~~ ~~set~~ a seq. of sets $P(n)$ $n \in \mathbb{N}$, ("n-ary operation symbols")

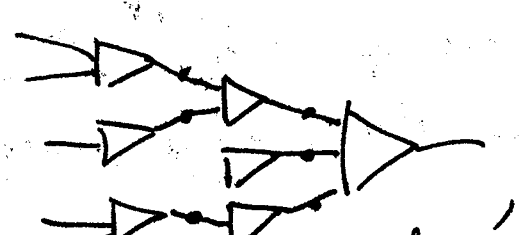
with maps $m_{k;n_1,\dots,n_k}: P(k) \times P(n_1) \times \dots \times P(n_k) \longrightarrow P(\sum n_i)$

Another way to generalise: this, in any tensor cat. $(\mathcal{C},\otimes)$ e.g. topological, linear, etc. monoids

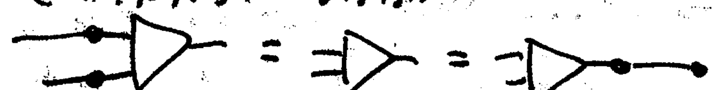


& a distinguished $e \in P(1)$

satisfying "assoc. & unit axioms":



$((h_1^1, h_1^2), (), (h_3^1))$ (g_1, g_2, g_3) f



$TX = (X)^* = \coprod_n X^n$, the free monoid on X, the set of words in X.

All about free monoids! Spiel from overhead ~~this~~ It's a set $P \xrightarrow{a} T1$ "arity"

$\coprod_{n \in \mathbb{N}} P(n)$ $\mathbb{N}$,

with maps $(TP)(k) \times P(k) \longrightarrow P(\dots)$? Maps m & e over $T1$:

"words in P of length k"

$$\begin{tikzcd}
& & \cdot \arrow[dl] \arrow[dr] \arrow[rr, "m"] & & P \arrow[dddlll, bend left, "a"] \\
& TP \arrow[dl, "Ta"'] \arrow[dr, "T!\ \text{"lengths"}"] \arrow[ddl, bend right, "\text{"sum of arities"}"'] & & P \arrow[dl] & \\
TT1 \arrow[d, "\mu\ (+)"'] & & T1 & & \\
T1 & & & &
\end{tikzcd}
\qquad \& \qquad
\begin{tikzcd}
1 \arrow[rr, "e"] \arrow[dr, "\eta"'] & & P \arrow[dl, "a"] \\
& T1 &
\end{tikzcd}$$

~~where~~ satisfying axioms

$$\begin{tikzcd}
TP \times_{T1} TP \times_{T1} P \arrow[r, "(\mu \times m)"] \arrow[d, "(Tm \times 1_P)"'] & TP \times_{T1} P \arrow[d] \arrow[dl, phantom, "//"] \\
TP \times_{T1} P \arrow[r] & P
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& & & TP \times_{T1} TP \times_{T1} P \arrow[dl] \arrow[dr] & & \\
& & \cdot \arrow[dl] \arrow[dr] & & \cdot \arrow[dl] \arrow[dr] & \\
& T^2P \arrow[dl] \arrow[drr] & & TP \arrow[d] \arrow[dr] & & P \arrow[dl] \\
T^3 1 \arrow[d] & & & T^2 1 & T1 & \\
T^2 1 \arrow[d] & & & & & \\
T1 & & & & &
\end{tikzcd}$$

(assoc) (& unit similar)

"As an operad is indexed by the free monoid on one ~~object~~ gen., it can compose words in its own elts, the behaviour of arities of comp'n & the unit are describable in terms of 1 & $(n_1, \ldots, n_k) \longmapsto \sum n_i$, i.e. the inj'n of generators & the "multiplication" of $T1 = \mathbb{N}$

$$1 \xrightarrow{\eta} T1 \xleftarrow{\mu} T^2 1,$$

the assoc. & unit axioms are described in terms of words of words from P unbracketing these to make words etc...

i.e. the monad structure $P \xrightarrow{\eta} TP \xleftarrow{\mu} T^2 P$ "

→ So... now, just as this gave a weakened notion of monoid*, what if we replace "monoid" with "ω-category" throughout? I.e.

→ **Def'n:** Let $\mathcal{E} = [\mathbb{G}^{op}, \underline{\text{Sets}}]$, the category of globular sets ($X_0 \overset{s}{\underset{t}{\leftleftarrows}} X_1 \overset{s}{\underset{t}{\leftleftarrows}} X_2 \leftleftarrows \ldots$), $T : \mathcal{E} \to \mathcal{E}$ the "free strict ω-category monad". Then a globular operad is defined by diagrams exactly as those above. WHAT THE HELL'S GOING ON??

[* Big difference: there we needed to go topological to talk about contractibility; here we won't.]

→ Well — recall: a <u>monad</u> on a cat $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$,

with nat. transf. $T^2X \xrightarrow{\mu_X} TX \xleftarrow{\eta_X} X$

satisfying certain axioms. It abstracts the situation

"take the free widget FX on a set X, then forget its widget struc so you've just got a set."

η is "injection of generators", μ is "unbracketing".

e.g. the <u>free monoid</u> monad on sets — you know well.

→ e.g. the <u>free ω-category</u> monad T on $\hat{\mathbb{G}} = [\mathbb{G}^{op}, Sets]$.

<u>Now</u>: for a glob. set X, cells of TX:

<u>pastings diags</u> of cells from X e.g.

$$\begin{tikzcd} x \arrow[r, bend left=70, "f"] \arrow[r, "f'" description] \arrow[r, bend right=70, "f''"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & z \end{tikzcd} \in TX(2)$$

μ_X is "unbracketing": pasting pasting diags together;
η_X is "injection of generators"

where $x, y, z \in X(0)$, f etc, $s(f) = x$, $t(f) = y$, etc.

→ <u>So</u> what about $T1$? This replaces $\mathbb{N}$ in the defn of "globular operad", i.e. gives the <u>arities</u> of a glob. op. Here 1 is the terminal glob. set, with one cell in each dim'n. Cells of $T1$: <u>pasting diagrams</u> like those above, but <u>don't need to label</u>.

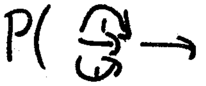
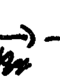
Also note: have <u>degenerate</u> ones, e.g. have not just a 1-cell $\xrightarrow{f}\xrightarrow{g}$ in $TX(1)$, but a 2-cell $\xrightarrow{f}\xrightarrow{g}$ in $TX(2)$, from the identity 2-cell on this cell, also have n-cells for higher n.

[There are various combinatorial representations of these pasting diags]

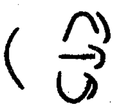
<u>So</u> a glob. operad starts with a glob. set $P \longrightarrow T1$,

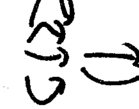
i.e. sets $P(\pi)$ for each p.d. π, "π-ary operations" "ways of composing diags of shape π".

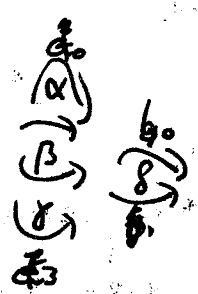
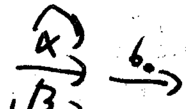
s.t. $f \in P(\pi)$ has $s(f) \in P(s(\pi))$, $t(f) \in P(t(\pi))$, "ways of composing the source & target"

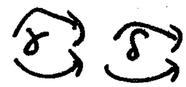
→ What's composition? e.g. Given $f \in P($  $)$, $t(f) = s(g) \in P($  $)$

$g \in P($  $)$,

$h \in P($  $)$,

should have a composite $h \cdot (g . f) \in P($  $)$

(since, intuitively, given  , compose  using f,

 using g,

then the results fit together as $s(g) = t(f)$, so can compose them using h).

Unwrap: this is exactly what the diagram for m says.

Similarly, have unit n-cells $e_n \in P(n\text{-globe})$, "composing a single n-cell to give itself."

→ Assoc, unit laws: discuss if time.

Talk 14.x.08

→ Exples: $P(\pi) = 1$, each π, gives the terminal glob. operad.

"Algebras for this" — haven't defined yet: — strict ω-cats
(= glob. set with elts of P interpreted as actual operations)   (Just like in "monoid" case.)

→ P a plain operad. Then have glob. op. "ΣP", "suspension",

$$\Sigma P(\bullet) = 1,$$

$$\Sigma P(\underbrace{\to\to\cdots\to}_{n}) = P(n)$$

$$\Sigma P(\pi) = \phi \text{ o/wise.}$$

→ X a glob set. "$\mathrm{End}(X)$". In plain case, had $\mathrm{End}(X)(n) = [X^n, X]$, for a set X.

Here... well, define X^{π} by

e.g. $X^{\bullet \Downarrow \bullet \Downarrow \bullet}$
$$= \left\{ \text{diags } \begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & z \end{tikzcd} \right\}$$

$= \{ \alpha, \beta \in X(2),\ f, \ldots, g' \in X(1),\ x, y, z \in X(0) :$
$s(\alpha) = f$, etc.$\}$

$$= \varprojlim \left(\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "t"] & & X_1 \arrow[dl] \arrow[dr] & \\ X_0 & X_2 \arrow[u] \arrow[d] & X_0 & X_2 \arrow[u] \arrow[d] & X_0 \\ & X_1 \arrow[ul] \arrow[ur] & & X_1 \arrow[ul] \arrow[ur] & \end{tikzcd} \right)$$

$\mathrm{End}(X)(\pi)$: not quite just $[X^{\pi}, X_n]$,
but coherent families of maps from $[X^{\pi}, X_n]$, $[X^{s(\pi)}, X_{n-1}]$, $[X^{t(\pi)}, X_{n-1}]$, ...
i.e. ways of composing a diag. of shape π & all its sources & tgts. in lower

→ Lets us define **algebras** — and not just in **Sets**, but in $[\mathbb{G}^{op}, \mathcal{C}]$, $\mathcal{C}$ with f. lims!

<u>Sketch</u>:

type theory case!

~~[illegible]~~ $\mathrm{End}(X_\bullet)$ in $\mathbb{C}$ [illegible] T [illegible])

T a type theory, X a type,

$X_\bullet$ the glob object

$$X \leftleftarrows X, X, \mathrm{Id}_X \leftleftarrows X, X, \mathrm{Id}_X, \mathrm{Id}_X, \mathrm{Id}^2_X, \leftleftarrows$$