→ Last time: type theory, Id types

What do types "look like"?
What struc's do their terms give?

Set: $\{\Gamma \vdash \tau : A\} \big/ \exists \pi \; \Gamma \vdash \pi : Id(\tau, \tau')$

Set with equiv. rel'n? Better:
Groupoid (= cat with inverses):

$\{\tau : A\} \big/ \tau = \tau'$,

$\{\pi : Id(\tau, \tau')\} \big/ \exists \sigma : Id(\pi, \pi')$

Reminiscent of: ——

→ Homotopy theory:

X (nice) top. space

Set $\pi_0(X)$ = {path cpts} = {points $x \in X$} $\big/ \exists p : x \to x'$

Groupoid $\pi_1(X)$ = $\begin{cases} \{\text{points } x \in X\} \\ \{\text{paths } x \xrightarrow{p} x'\} \big/ \text{homotopy} \end{cases}$

Weak 2-groupoid {pts}, {paths}, {htpies btw paths} / {htpy}

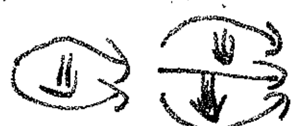
⋮

Weak ω-gpoid: never stop, never quotient!

Interesting struc: cts all info re htpy, & hence homology, of space ... at least.

Weak ω-gpoid:

$x : X$
$\pi : Id(\tau, \tau')$
$\sigma : Id(\pi, \pi')$
$\rho : Id(\rho, \rho')$
⋮



Homotopy: maps $Y \underset{g}{\overset{f}{\rightrightarrows}} X$, $f, g \in [Y, X]$

Homotopy: $H : I \to [Y, X]$, $H(0) = f$, $H(1) = g$. Or: $H : I \times Y \to X$

Rel endpts

Axiomatizing such strucs: HDCT!

~~Lots~~ ~~of~~ ~~operations~~

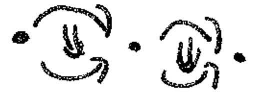
Set of n-cells, X_n, $n \in \mathbb{N}$,

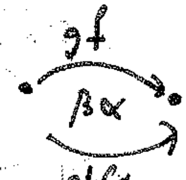
~~some~~ each n-cell has source & target $(n-1)$-cells, parallel:

"a globular set" $X_0 \underset{t}{\overset{s}{\leftleftarrows}} X_1 \underset{t}{\overset{s}{\leftleftarrows}} X_2 \underset{t}{\overset{s}{\leftleftarrows}} X_3 \leftleftarrows$

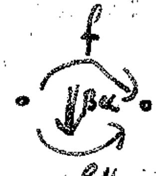
$ss = st$
$ts = tt$

Compositions:  $\bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet \quad \longmapsto \quad \bullet \xrightarrow{gf} \bullet$,





"$\beta \circ_0 \alpha$"



"$\beta \circ_1 \alpha$"

Unit maps, ...

s.t. ... etc. "can compose n-cells along a bounding k-cell, $k < n$"

$\to$ <u>Enriched</u> approach: a 2-<u>category</u> is:

set of objects ob $\mathcal{C}$

for $X, Y \in \text{ob}\,\mathcal{C}$, a "hom-category" $\mathcal{C}(X,Y)$,

& ~~functors~~ maps, $1 \longrightarrow \mathcal{C}(X,X) \qquad \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z)$

satisfying assoc. & unit axioms.

"A category enriched in Cat". [Enrichment: ~~general~~ much wider more general than this.]

So ~~bits~~ unwinding: objects, $\bullet X$

1-cells: objects of the hom categories, $X \xrightarrow{f} Y$

2-cells: maps of the - - - - -
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & Y \end{tikzcd}$$

compn: $\begin{tikzcd} X \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & Y \end{tikzcd}$ from comp'n in $\mathcal{C}(X,Y)$. $X \longrightarrow Y \longrightarrow Z$

$X \to Y \to Z$ from ob. part of $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z)$,

$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Z \end{tikzcd}$ .. morph. - - . . . - - - - - -

We're on the right track it seems...

→ Inductively: an $(n+1)$-cat. is a cat. enriched in n-Cat.

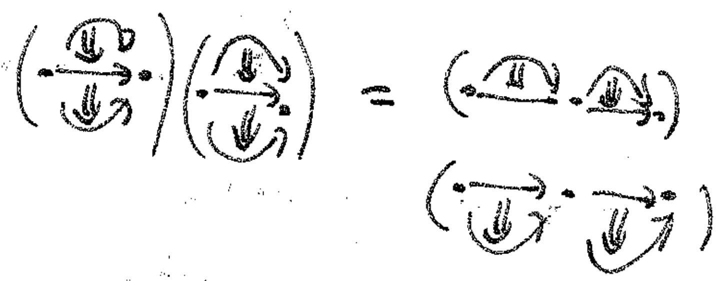
→ An ω-Cat is the "limit" of this process.

Good? Ish. Has cells as we pictured above,

& comp'n, unit maps,

but associativity & unit laws are strict:
& interchange

$$(\bullet \longrightarrow \bullet \longrightarrow \bullet) \longrightarrow \bullet \;=\; \bullet \longrightarrow (\bullet \longrightarrow \bullet \longrightarrow)$$



We want these only "up to homotopy", i.e. up to cells of the next dimension.

These strict n-categories: easy to define, not so many kinds of nat'l examples.

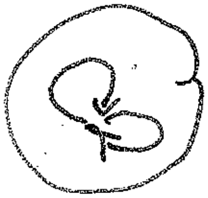
Weak n-cats: hard to define; lots of nat'l examples, of lots of different flavours.

(e.g., from spaces as above;

or eg: "weakly" associative - up to canonical "iso" $\otimes$ structures, e.g. any cat. you can think of with $\otimes$ or $\times$ or $\oplus$ or ... as one-object wk. 2-cat)

How do we weaken?

Warm-up: weaken the idea of a monoid.

e.g. loop space.  ~~$M(X,x)$~~ $\Omega(X,x)$

~~homotopy~~ canonical ~~homotopy~~ paths

$(fg)h \longrightarrow f(gh)$

canonical homotopy:

$$\begin{tikzcd} & (fg)(hk) \arrow[dr] & \\ ((fg)h)k \arrow[ur] \arrow[d] & & f(g(hk)) \\ (f(gh))k \arrow[r] & f((gh)k) \arrow[ur] & \end{tikzcd}$$

Well: first: these are really from

a path in the space of ternary operations,

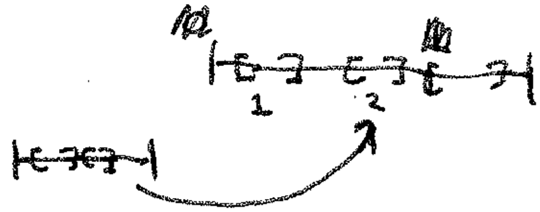
a homotopy in the space of 4-ary operations.

So: operads find a space $P(n)$ of "n-ary operations", each n, P "closed under composition", & each $P(n)$ contractible.

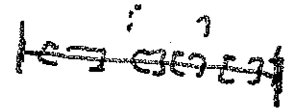
In this case, ~~$P(n)$ = "~~

P = "little 1-discs":

$P(n)$ = (collns of n disjt closed sub-intervals of $[0,1]$.)



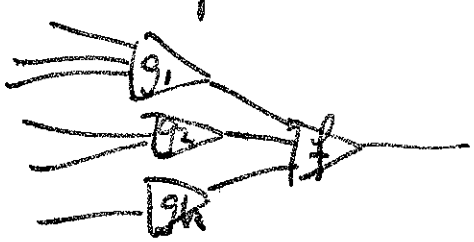
Compose to get



Axiomatic: a topological operad

is spaces $P(n)$ $n \in \mathbb{N}$,

composition maps $P(n_1) \times \dots \times P(n_k) \times P(k) \to P(\sum n_i)$



an identity elt $1 \in P(1)$,

satisfying unit & assoc. laws .... (not hard to write down)

→ Exple: X a ~~nice~~ space. $\mathrm{End}(X)(n) = [X^n, X]$ map space.

Def'n an action of P on X is a map of operads $P \to \mathrm{End}\,X$, i.e. for each $p \in P(n)$, a map $f: X^n \to X$.

"univ. alg. in a top. setting" p6.

<u>or</u> "a one object multicat".