Talk, 30.ix.08

→ ~~Objects;~~

DTT.

Recall: type theory.

Form of judgments. $x : X,\ y : Y \vdash \tau : Z$

X, Y, Z types, & τ a term built up.

"dependent": types themselves can depend on the context,

e.g. ~~the classical type~~

$n : \mathbb{N} \vdash A^n$ type.

constructors such as "π-types", "Σ-types, Id-types" — type indexed products,

$$\frac{\Gamma,\ x : A \vdash B(x)\ \text{type}}{\Gamma \vdash \prod_{x:A} B(x)\ \text{type}} \quad (\pi\text{-form})$$

$$\frac{\Gamma,\ x : A \vdash \tau(x) : B(x)}{\Gamma \vdash \lambda x.\tau(x) : \prod_{x:A} B(x)} \quad (\pi\text{-intro})$$

$$\frac{\Gamma \vdash f : \prod_{x:A} B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash f \cdot a : B(a(x))} \quad (\pi\text{-elim})$$

$$\frac{\Gamma, x:A \vdash \tau(x) : B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x.\tau(x)) \cdot a = \tau(a) : B(a)} \quad (\pi\text{-comp.})$$

optional.

$$\frac{\Gamma \vdash g, f : \prod_{x:A} B(x) \qquad \Gamma, x : A \vdash f \cdot x = g \cdot x : B(x)}{\Gamma \vdash f = g : \prod_{x:A} B(x)} \quad (\pi\text{-ext}).$$

stronger:
$$\frac{\Gamma \vdash f : \prod_{x:A} B(x)}{\Gamma \vdash (\lambda x : A.\ f x) = f : \prod_{x:A} B(x)} \quad (\pi\eta) \quad \text{— Stronger!}$$

Can model this in <u>Sets</u> or a sufficiently set-like category—!

~~Basically~~ roughly, a <u>locally cartesian closed category</u>. (modulo technicalities)

Essentially, ~~these~~ 'types look like sets'.

But— this was ext<u>ensional</u>! Would like to record that f, g could agree on all values, but ~~to~~ be inten<u>sionally</u> different. <u>So</u> — use $=$ for the ~~weaker~~ finer notion of "definitional"/"intensional" equality. ~~Decidable~~ New notion: provable equality within the type theory — "propositional"/"extensional" equality: a <u>type</u> of "proofs of identity between x & y":

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x, y : A \vdash \mathrm{Id}_A(x,y) \text{ type}} \ \text{Id-form}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \vdash r(x) : \mathrm{Id}_A(x,x)} \ \text{Id-intro}$$

$$\frac{\Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash C(x,y,p) \text{ type} \qquad \Gamma, x : A \vdash d(x) : C(x,x,r(x))}{\Gamma, x, y : A, p : \mathrm{Id}(x,y) \vdash J_d(x,y,p) : C(x,y,p)} \ \text{Id-elim}$$

$$\frac{\text{(hyps as in Id-elim)}}{\Gamma, x : A \vdash J_d(x,x,r(x)) = d(x) : C(x,x,r(x))} \ \text{Id-comp}$$

<u>Slogan</u>: "If you can do something using $r(x)$, you can do it using <u>any</u> proof of identity!"

Play around: derive <u>transitivity</u> of equality:

want

use $d(x) = \lambda \cancel{z}, q . q$

$$\dfrac{\dfrac{\dfrac{}{x : A \vdash d(x) : \prod_{z:A} \prod_{q:\mathrm{Id}(x,z)} \mathrm{Id}(x,z)}}{\cancel{x, p: \mathrm{Id}(x,y),}\ x,y:A\ \ p:\mathrm{Id}(x,y) \vdash t(p,-) : \prod_{z:A} \prod_{q:\mathrm{Id}(y,z)} \mathrm{Id}(x,z)}}{x,y,z:A,\ p:\mathrm{Id}(x,y),\ q:\mathrm{Id}(y,z) \vdash t(p,q) : \mathrm{Id}(x,z)} \ (\mathrm{struc}, \Pi)$$

Notice: could have used Id-elim on p instead of q.

<u>But</u> — this ~~suggests~~ gives inkling of how we <u>can</u> use Id-types how we'd like to

So now we drop the π-ext rules, & consider this theory. What do types look like now, taking into account $\mathrm{Id}(x,y)$? i.e. have the set $\{\Gamma \vdash \tau : A\}$ / $\Gamma \vdash \tau = \tau' : A$)

& extra struc from Id_A. At least: a set with an equiv. relation:

"$\tau \sim \tau'$ if $\exists\, \sigma$ $\Gamma \vdash \sigma : \mathrm{Id}_A(\tau, \tau')$".

→ But — keep track not just of "is $\mathrm{Id}(\tau,\tau')$ inhabited?", but of all its terms;

i.e. want a structure keeping track of a set of things, & a set of ways to show they're equal:

a GROUPOID. Recall: a groupoid is a category where all morphisms are isomorphisms.

→ "A many-object group": group abstracts the situation "symmetries / ~~actions~~ of a single object", groupoid of "isos between a bunch of objects".

→ Fundamental exple: X a nice top. space, ~~loc. path-conn'd &~~

$\pi_0(X) = \{\text{path-cpts}\} = \{\text{points}\} / \text{homotopy paths}$

Fundamental groupoid (extending fund'l group) $\Pi_1(X) = \{\text{points}, \text{paths}\} / \text{htpies}$

Theorem: $\Gamma \vdash A$ type, $\Gamma \vdash \Delta$ ctxt. Then glob set

$X_n := \{\Gamma, \Delta \vdash \tau : \Sigma x, y.\, A, \mathrm{Id}(x,y), \ldots\} / =$

is a weak ω-gpoid.