ULTRAPRODUCTS AND CONTINUOUS FAMILIES OF MODELS

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Marmolejo

Signature of Author

De aquel lado, a Cristina y Juan, mis padres.

To Karen on this side.

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Abstract

Let $\boldsymbol{P}$ be a small pretopos. Makkai showed that the pretopos (i.e. the language) can be recovered from the category of models of the pretopos (i.e. $\boldsymbol{Set}$-valued functors preserving the pretopos structure). The realization that ultraproduct functors can be expressed as composition of functors on categories of sheaves over topological spaces opens the door for using continuous families of models, that is, categories indexed over topological spaces.

We introduce a special kind of category indexed over topological spaces in which it is possible to define ultraproduct functors. This involves continuous functions $f : Y \to X$ for which the functors $f_* : Sh(Y) \to Sh(X)$ preserve the pretopos structure. We give a characterization of such functions. Each of these indexed categories produces a pre-ultracategory in the sense of Makkai.

We also consider the 2-adjunction $\boldsymbol{PRETOP}^{op} \underset{\boldsymbol{Mod}(\_)}{\overset{\boldsymbol{Set}^{(-)}}{\leftrightarrows}} \boldsymbol{CAT}$ and the 2-monad it generates. We show that each algebra for this 2-monad carries a pre-ultracategory structure as well. We induce another 2-monad over the category of algebras and show that these new algebras carry the structure of ultracategories.

We combine both approaches by defining a 2-adjunction over the 2-category of special indexed categories mentioned above and show that the corresponding algebras also carry ultracategory structures.

Finally, aiming at giving filtered colimits a bigger role in the picture we generalize a theorem of Lever, namely, that indexed functors from the indexed category that has the category of sheaves $Sh(X)$ over the topological space X, to itself is equivalent to the category of filtered colimit preserving functors from $\boldsymbol{Set}$ to itself.

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Introduction

The concept of pretopos was introduced by Grothendieck in [1] in relation with coherent toposes. A pretopos is a category with finite limits, strict initial object, stable disjoint finite coproducts and stable quotients of equivalence relations. Functors between pretoposes that preserve the pretopos structure are called elementary. Smallness is also required in [1] but we allow our pretopos to be "big", so for example the category ***Set*** of sets is a pretopos. Makkai and Reyes in [18] study the relation between coherent theories and pretoposes. They show there how to construct a small pretopos for any coherent theory that essentially codifies the information of the theory in the sense that the category of models for the coherent theory and the category of elementary functors from the pretopos are equivalent. That is we can replace the theory by the pretopos. The construction of the pretopos involves as a first step the construction of a logical category. A category is logical if it has finite limits, stable finite sups of subobjects and stable images. This logical category can also replace the theory, however there are two good reasons to use pretoposes instead of logical categories. The first one is that there is a criteria to determine whether an elementary functor between pretoposes is an equivalence (see 7.1.8 in [18] or Lemma 1.15 below). The second reason is the so called conceptual completeness: If an elementary $F : \boldsymbol{P} \to \boldsymbol{Q}$ between pretoposes induces by composition an equivalence $\boldsymbol{Mod}(\boldsymbol{Q}) \to \boldsymbol{Mod}(\boldsymbol{P})$ then F is an equivalence (see 7.1.8 in [18] or Theorem 1.16 below). Here $\boldsymbol{Mod}(\boldsymbol{P})$ denotes the category of elementary functors from $\boldsymbol{P}$ to ***Set***. There are some questions to be asked in this context. One is whether it is possible to recover the language from the category of models. Another one is under what conditions a category is a category of models. On the one hand we want to recover the pretopos $\boldsymbol{P}$ from the category $\boldsymbol{Mod}(\boldsymbol{P})$ and on the other we want to find conditions on a category $\boldsymbol{A}$ for it to be of the form $\boldsymbol{Mod}(\boldsymbol{P})$ for some small pretopos $\boldsymbol{P}$. This resembles for example the well known Gabriel-Ulmer duality (see [17]) in which we have equivalences $\boldsymbol{C} \to \boldsymbol{LFC}(\boldsymbol{LEX}(\boldsymbol{C}, \boldsymbol{Set}), \boldsymbol{Set})$ for any small left exact category $\boldsymbol{C}$ and $\boldsymbol{A} \to \boldsymbol{LEX}(\boldsymbol{LFC}(\boldsymbol{A}, \boldsymbol{Set}), \boldsymbol{Set})$ for any locally finitely presentable category $\boldsymbol{A}$ where $\boldsymbol{LEX}$ denote the category of left exact categories in the second universe and $\boldsymbol{LFC}$ is the category of categories with small limits and small filtered colimits in the second universe. Makkai in [15] proves one half of the above duality for pretoposes. Notice first that in the equivalence $\boldsymbol{C} \to \boldsymbol{LFC}(\boldsymbol{LEX}(\boldsymbol{C}, \boldsymbol{Set}), \boldsymbol{Set})$ what is done is to consider functors $\boldsymbol{LEX}(\boldsymbol{C}, \boldsymbol{Set}) \to \boldsymbol{Set}$ and add conditions on them (the $\boldsymbol{LFC}$ part) to cut down to the ones that are of the form ev_C for some C in $\boldsymbol{C}$. For pretoposes we have to replace $\boldsymbol{LEX}(\boldsymbol{C}, \boldsymbol{Set})$ by $\boldsymbol{Mod}(\boldsymbol{P})$. $\boldsymbol{Mod}(\boldsymbol{P})$ has filtered colimits and they are calculated as in $\boldsymbol{Set}^{\boldsymbol{P}}$. However we can not in general guarantee the existence of any other kind of limits or colimits. What can be used is another construction that is also pointwise, namely the ultraproduct construction. Ultraproducts are mixed limits (filtered colimits of products) and therefore have very few canonical arrows, as opposed to honest limits or colimits. In [15] the part corresponding to $\boldsymbol{LFC}$ is taken by ultracategories. An ultracategory is obtained in two steps. First a pre-ultracategory is a category $\boldsymbol{A}$ together with a functor $[\mathcal{U}] : \boldsymbol{A}^I \to \boldsymbol{A}$ for every ultrafilter $(I,\mathcal{U})$. Functors between them have transition isomorphism relating the corresponding functors of the form $[\mathcal{U}]$. The concept of ultramorphism is introduced to supply enough arrows to and from ultraproducts. An ultracategory is a pre-ultracategory together with ultramorphisms. This suffices to prove an equivalence of the form $\boldsymbol{P} \to \boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ where $\boldsymbol{UC}$ denotes the category of ultracategories in [15]. The other side of the question is still open.

The idea that started this paper is that we can recover the ultraproduct functor $[\mathcal{U}] : \boldsymbol{Set}^I \to \boldsymbol{Set}$ for every ultrafilter $(I,\mathcal{U})$ using categories of sheaves. Specifically the functor $[\mathcal{U}]$ is naturally equivalent to the composition $\boldsymbol{Set}^I \xrightarrow{\simeq} Sh(I) \xrightarrow{\mu_*} Sh(\beta I) \xrightarrow{\mathcal{U}^*} \boldsymbol{Set}$ where βI is the Stone-Čech compactification of I, $\mu : I \to \beta I$ is the usual embedding and $\mathcal{U}^*$ is the functor associated to the continuous function $\mathcal{U} : 1 \to \beta I$

that picks the ultrafilter $\mathcal{U} \in \beta I$. So we consider categories indexed over the category $\boldsymbol{Top}$ of topological spaces and continuous functions. We follow Paré and Schumacher [19], the approach in Benabou [3] is via fibrations. A $\boldsymbol{Top}$-indexed category $\mathcal{A}$ consists of a category $\mathcal{A}^X$ for every topological space X and a functor $f^* : \mathcal{A}^X \to \mathcal{A}^Y$, for every continuous function $f : Y \to X$ subject to some coherence conditions. In particular if we take the category $Sh(X)$ for every topological space X and the usual $f^* : Sh(X) \to Sh(Y)$ we obtain a $\boldsymbol{Top}$-indexed category that we denote by $\mathcal{SET}$. This category plays the rôle of sets in $\boldsymbol{Top}$-indexed categories. $f^* : Sh(X) \to Sh(Y)$ is left exact and has a right adjoint. Thus f^* is elementary. We can define then, for every pretopos $\boldsymbol{P}$ the $\boldsymbol{Top}$-indexed category of models of $\boldsymbol{P}$. We take the category $\boldsymbol{Mod}_{Sh(X)}(\boldsymbol{P})$ for every space X and define $f^* : \boldsymbol{Mod}_{Sh(X)}(\boldsymbol{P}) \to \boldsymbol{Mod}_{Sh(Y)}(\boldsymbol{P})$ by composition with $f^* : Sh(X) \to Sh(Y)$ for every continuous $f : Y \to X$, where $\boldsymbol{Mod}_{Sh(X)}(\boldsymbol{P})$ denotes the category of elementary functors from $\boldsymbol{P}$ to $Sh(X)$. We denote this $\boldsymbol{Top}$-indexed category by $\mathcal{MOD}(\boldsymbol{P})$. To be able to recover the ultraproduct functors we have to take into account the functors of the form μ_* as above. For this purpose we introduce the concept of ultrafinite function: A continuous function $f : Y \to X$ is called ultrafinite if the functor $f_* : Sh(Y) \to Sh(X)$ is elementary. Notice that for an ultrafinite f the functor $f^* : \boldsymbol{Mod}_{Sh(X)}(\boldsymbol{P}) \to \boldsymbol{Mod}_{Sh(Y)}(\boldsymbol{P})$ has a right adjoint. Furthermore we recover the ultraproduct functors $[\mathcal{U}] : \boldsymbol{Mod}(\boldsymbol{P})^I \to \boldsymbol{Mod}(\boldsymbol{P})$ as the composition $\boldsymbol{Mod}(\boldsymbol{P})^I \xrightarrow{\simeq} \boldsymbol{Mod}_{Sh(I)}(\boldsymbol{P}) \xrightarrow{\mu_*} \boldsymbol{Mod}_{Sh(\beta I)}(\boldsymbol{P}) \xrightarrow{\mathcal{U}^*} \boldsymbol{Mod}(\boldsymbol{P})$. Accordingly we characterize those continuous functions that are ultrafinite and restrict to $\boldsymbol{Top}$-indexed categories for which f^* has a right adjoint f_* for every ultrafinite f. Functors between these are those that behave nicely with these adjoints. We denote this category by $\mathfrak{Los}$. With the category $\mathfrak{Los}$ we can recover the pre-ultracategory structure but unfortunately it is not enough to recover the general ultramorphisms.

There is another way to recover the pre-ultracategory structure via algebras over $\boldsymbol{CAT}$, and with a monad over these algebras we can also recover the ultramorphisms. Consider the 2-monad $\boldsymbol{T}$ generated by the 2-adjunction $\boldsymbol{PRETOP}^{op} \underset{\boldsymbol{Mod}(\_)}{\overset{\boldsymbol{Set}^{(-)}}{\rightleftarrows}} \boldsymbol{CAT}$.

We can define a functor $\boldsymbol{T}\text{-}ALG \to \boldsymbol{PUC}$ where $\boldsymbol{T}\text{-}ALG$ denotes the 2-category of $\boldsymbol{T}$-algebras and $\boldsymbol{PUC}$ denotes the 2-category of pre-ultracategories. We obtain another 2-adjunction $\boldsymbol{PRETOP}^{op} \underset{(\boldsymbol{Mod}(\_),\Phi_{(\_)})}{\overset{\boldsymbol{T}\text{-}ALG(\_,(\boldsymbol{Set},\Psi))}{\leftrightarrows}} \boldsymbol{T}\text{-}ALG$ where $\Phi_{\boldsymbol{P}}$ and Ψ are $\boldsymbol{T}$-algebra structures we define below. Let $\boldsymbol{S}$ denote the 2-monad generated by this adjunction. We can define then a 2-functor $\boldsymbol{S}\text{-}ALG \to \boldsymbol{UC}$ where $\boldsymbol{UC}$ denotes the 2-category of ultracategories.

Our proofs about algebras are based on the following observation. Suppose we have functors $H : \boldsymbol{A} \to \boldsymbol{B}$, $R : \boldsymbol{B} \to \boldsymbol{A}$ and a natural transformation $\theta : RH \to 1_{\boldsymbol{A}}$. If $\boldsymbol{B}$ has a functorial weak initial object then $\boldsymbol{A}$ has a functorial weak initial object as well. A functorial weak initial object is a weak initial object with a functorial choice of arrows from it to any other object. When the natural transformation θ is an isomorphism, the existence of functorial weak colimits in $\boldsymbol{B}$ implies the existence of functorial weak colimits in $\boldsymbol{A}$. It is well known that colimits exist if the category has functorial weak colimits and split idempotents. In this context it is easy to see that $\boldsymbol{A}$ has split idempotents if $\boldsymbol{B}$ does.

The above setting is specially well suited for algebras over a 2-monad. If we have a 2-monad $\boldsymbol{T} = (T, \eta, \mu)$ over $\boldsymbol{CAT}$ for example and a strict algebra $(\boldsymbol{A}, \Phi)$ then one of the diagrams for Φ is

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "\eta \boldsymbol{A}"] \arrow[dr, "1_{\boldsymbol{A}}"'] & T\boldsymbol{A} \arrow[d, "\Phi"] \\ & \boldsymbol{A} \end{tikzcd}$$

If $T\boldsymbol{A}$ is a 'good' category then $\boldsymbol{A}$ will necessarily inherit some of the good properties of $T\boldsymbol{A}$. In particular the existence of certain kinds of limits or colimits. Furthermore, the other commutative diagram for algebras will tell us how to calculate these limits and colimits on $\boldsymbol{A}$: Simply take the diagram over $\boldsymbol{A}$, compose with $\eta\boldsymbol{A}$, calculate the limit or colimit in $T\boldsymbol{A}$ and apply Φ. For example consider the 2-monad given by the 2-adjunction $\boldsymbol{Set}^{(-)} \dashv \boldsymbol{Set}^{(-)} : \boldsymbol{CAT}^{op} \to \boldsymbol{CAT}$. In this case having an algebra structure on a category $\boldsymbol{A}$ implies that $\boldsymbol{A}$ is complete and cocomplete. We note here that there are some size problems to be resolved.

One way of trying to settle these size problems and at the same time give a good framework in which to attempt a solution to the second problem (namely characterizing those categories that are of the form $\boldsymbol{Mod}(\boldsymbol{P})$) is to combine the last two approaches. That is, we define a 2-adjunction $\boldsymbol{PRETOP}^{op} \underset{\mathcal{MOD}(\_)}{\overset{\mathfrak{Cos}(\_,\mathcal{SET})}{\leftrightarrows}} \mathfrak{Cos}$, generate the corresponding 2-monad $\boldsymbol{T}$ and define a functor $\boldsymbol{T}\text{-}ALG \to \boldsymbol{UC}$.

Finally, in a closely related development we generalize a theorem of Lever [11]. Lever showed that there was an equivalence between the categories $\boldsymbol{Filt}(\boldsymbol{Set}, \boldsymbol{Set})$ of filtered colimit preserving functors from $\boldsymbol{Set}$ to $\boldsymbol{Set}$ and $\boldsymbol{Top}\text{-}ind(\mathcal{SET}, \mathcal{SET})$ of $\boldsymbol{Top}$-indexed functors from $\mathcal{SET}$ to $\mathcal{SET}$. We define a $\boldsymbol{Top}$-indexed category $\mathcal{A}$ for every category $\boldsymbol{A}$ with filtered colimits and products by taking coalgebras over $\boldsymbol{A}^{|X|}$ for every topological space X and show that we get an equivalence between $\boldsymbol{Filt}(\boldsymbol{A}, \boldsymbol{Set})$ and $\boldsymbol{Top}\text{-}ind(\mathcal{A}, \mathcal{SET})$. The definition of the cotriple is very similar to the one induced by the adjunction $Sh(X) \rightleftarrows \boldsymbol{Set}^{|X|}$. This will allow us to prove that whenever we have a $\boldsymbol{Top}$-indexed functor $F : \mathcal{MOD}(\boldsymbol{P}) \to \mathcal{SET}$ we have that the functor $F^1 : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ preserves filtered colimits.

The account chapter by chapter is as follows.

In chapter 1 we review the definition of pretopos and its relation to coherent toposes; we consider some properties of pretoposes we will need later, especially the ones concerning equivalence relations. We show that for any pretopos $\boldsymbol{P}$ and any object P in $\boldsymbol{P}$ the category $\boldsymbol{P}/P$ is a pretopos and that for any other pretopos $\boldsymbol{Q}$, the category $\boldsymbol{Mod_Q}(\boldsymbol{P}/P)$ is equivalent to the category whose objects are pairs (M, a) with M in $\boldsymbol{Mod}(\boldsymbol{P})$ and a a global element of MP. We use this description to give a categorical proof of the existence of an arrow into an ultrapower of another model under certain conditions. Finally we give a combinatorial description of the left adjoint to the forgetful functor $\boldsymbol{Pretop} \to \boldsymbol{Lex}$.

Chapter 2 is devoted to the concepts of ultracategory and ultramorphism. There we give a proof of Makkai's theorem (the equivalence of a small pretopos $\boldsymbol{P}$ and the category $\boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$. We follow Makkai's [15] in this chapter fairly closely.

In chapter 3 we consider categories indexed by topological spaces. We first review the concepts of indexed category theory drawing mainly from Paré and Schumacher [19] and also from Lever [11]. We then introduce the concepts of ultrafinite continuous function. The $\boldsymbol{Top}$-indexed categories that have right adjoints for the functors induced by ultrafinite functions are introduced next and are called Los categories. We close the chapter with a characterization of ultrafinite continuous functions.

In chapter 4 we start with a brief review of the folklore of functorial weak (co)limits. We then explore the relation between functorial weak (co)limits and retractions of categories. We apply these results to show that if a left exact category $\boldsymbol{C}$ has an algebra structure for the 2-monad generated by the adjunction $\boldsymbol{Pretop} \rightleftarrows \boldsymbol{Lex}$, then $\boldsymbol{C}$ is a pretopos. This points the way to show that the forgetful functor $\boldsymbol{Pretop} \to \boldsymbol{Lex}$ is monadic. Further analysis of this will have to await another paper. We again apply these results to show that algebras for different 2-monads over $\boldsymbol{CAT}$ have certain limits and colimits. We consider then in detail the two successive monads of pretoposes over $\boldsymbol{CAT}$ we are interested in and their relation with pre-ultracategories and ultracategories.

In chapter 5 we combine the approaches from chapters 3 and 4 by defining a monad over the category $\mathfrak{Los}$. We again relate this category of algebras with ultracategories.

In chapter 6 we define $\boldsymbol{Top}$-indexed categories of coalgebras over categories with filtered colimits and products. We generalize the result in Lever [11] and use this result to show that any $\boldsymbol{Top}$-indexed functor $F : \mathcal{MOD}(\boldsymbol{P}) \to \mathcal{SET}$ satisfies that $F^1 : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ preserves filtered colimits.

A Word About Size

We work in the setting of Grothendieck universes. That is we fix Grothendieck universes $\boldsymbol{U}_1 \in \boldsymbol{U}_2 \in \boldsymbol{U}_3$. Sets, pretoposes, categories in $\boldsymbol{U}_1$ are called small. The categories of small sets, small pretoposes, small categories are denoted by $\boldsymbol{Set}$, $\boldsymbol{Pretop}$, $\boldsymbol{Cat}$ respectively. We denote the category of sets in $\boldsymbol{U}_2$ by $\boldsymbol{SET}$, similarly $\boldsymbol{PRETOP}$ and $\boldsymbol{CAT}$ denote the categories (2-categories rather) of pretoposes and categories in the second universe $\boldsymbol{U}_2$ respectively. Then $\boldsymbol{Set}$ is an object in $\boldsymbol{SET}$. $\boldsymbol{SET}$ is not a category in $\boldsymbol{U}_2$ but it is a category in $\boldsymbol{U}_3$.

In this paper it is always assumed that limits and colimits are taken over diagrams with small domain.

Chapter 1

Pretoposes

1.1 Definition and Background

As we pointed out in the introduction the concept of pretopos comes from [1]. In this paper however we adopt the definition given in [15] that is equivalent except that the former definition asks for smallness.

Definition 1.1. The category $\boldsymbol{P}$ is a pretopos if and only if

1. $\boldsymbol{P}$ has finite limits.
2. $\boldsymbol{P}$ has a strict initial object.
3. $\boldsymbol{P}$ has stable disjoint finite coproducts.
4. $\boldsymbol{P}$ has stable quotients of equivalence relations.

A functor $F : \boldsymbol{P} \to \boldsymbol{Q}$ between pretoposes is called elementary if and only if it preserves finite limits, initial object, finite coproducts and quotients of equivalence relations.

If we denote the initial object by 0, it being *strict* means that for every P in $\boldsymbol{P}$, an arrow $P \to 0$ is necessarily an isomorphism.

Given objects $Q_1, .., Q_n$ in $\boldsymbol{P}$, the coproduct is *disjoint* if for every $j, k \in \{1, .., n\}$ $j \neq k$ implies that the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Q_j \arrow[d, "i_j"] \\ Q_k \arrow[r, "i_k"'] & \coprod_{k=1}^{n} Q_k \end{tikzcd}$$

is a pullback. Given $R \to \coprod_{k=1}^{n} Q_k$ in $\boldsymbol{P}$ we can form the pullback

$$\begin{tikzcd} P_k \arrow[r, "\pi_{k2}"] \arrow[d, "\pi_{k1}"'] & R \arrow[d] \\ Q_k \arrow[r, "i_k"'] & \coprod_{k=1}^{n} Q_k \end{tikzcd}$$

for every k. We say the coproduct is *stable* if the induced map

$$\coprod_{k=1}^{n} P_k \xrightarrow{\langle \pi_{k2} \rangle_k} R$$

is an isomorphism. It is not hard to see that, if the coproducts are disjoint and stable, then the injections into the coproduct are monomorphisms.

Given an equivalence relation $P \overset{f}{\underset{g}{\rightrightarrows}} Q$ in $\boldsymbol{P}$, a *quotient* for the equivalence relation is a coequalizer $Q \xrightarrow{r} R$ of f and g such that the square

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "f"'] & Q \arrow[d, "r"] \\ Q \arrow[r, "r"'] & R \end{tikzcd}$$

is a pullback. It is *stable* if the pullback of r along any arrow $A \to R$ is the quotient of some equivalence relation.

Given pretoposes $\boldsymbol{P}$ and $\boldsymbol{Q}$ we denote by $\boldsymbol{Mod_Q}(\boldsymbol{P})$ the category whose objects are elementary functors from $\boldsymbol{P}$ to $\boldsymbol{Q}$ and whose arrows are natural transformations between these. We call $\boldsymbol{Mod_Q}(\boldsymbol{P})$ the category of models of $\boldsymbol{P}$ in $\boldsymbol{Q}$. Clearly, the category $\boldsymbol{Set}$ is a pretopos and for any pretopos $\boldsymbol{P}$ we denote $\boldsymbol{Mod_{Set}}(\boldsymbol{P})$ simply by $\boldsymbol{Mod}(\boldsymbol{P})$.

Following the notation from [8] (that refers in its turn to [1]), a topos $\boldsymbol{E}$ is called *coherent* if it is equivalent to a category of the form $Sh(\boldsymbol{C}, J)$ for some site $(\boldsymbol{C}, J)$ with $\boldsymbol{C}$ a small left exact category and J generated by a pretopology in which every covering family is finite. An object X in a topos $\boldsymbol{E}$ is called *compact* if every epimorphic family $\{Y_i \to X\}_I$ with codomain X contains a finite epimorphic subfamily, X is called *stable* if, for any pair of arrows $S \to X \leftarrow T$ with S and T compact we have that the pullback $S \times_X T$ is compact, X is called *coherent* if it is both, compact and stable. We have (see 7.37 in [8])

Theorem 1.1. *If $\boldsymbol{E}$ is a coherent topos and $\boldsymbol{E}_{coh}$ is the full subcategory of $\boldsymbol{E}$ of coherent objects, then $\boldsymbol{E}_{coh}$ is an essentially small pretopos and the inclusion $\boldsymbol{E}_{coh} \to \boldsymbol{E}$ is elementary.* $\square$

Given a small pretopos $\boldsymbol{P}$ we can consider the precanonical topology J (J is generated by the pretopology whose covering families are all finite epimorphic families). We have (see 7.40 in [8])

Theorem 1.2. *A topos $\boldsymbol{E}$ is coherent if and only if there exists a small pretopos $\boldsymbol{P}$ such that $\boldsymbol{E}$ is equivalent to the category $Sh(\boldsymbol{P}, J)$ where J is the precanonical topology on $\boldsymbol{P}$. Furthermore, the pretopos $\boldsymbol{P}$ is determined up to equivalence by $\boldsymbol{E}$.*
$\square$

The pretopos $\boldsymbol{P}$ determined by a coherent topos $\boldsymbol{E}$ is of course $\boldsymbol{E}_{coh}$. From 7.45 and 7.47 in [8] we have

Theorem 1.3. *If $\boldsymbol{P}$ is a small pretopos, J the precanonical topology on $\boldsymbol{P}$ and M_0 the elementary functor $M_0 = (\boldsymbol{P} \to (Sh(\boldsymbol{P}, J))_{coh} \to Sh(\boldsymbol{P}, J))$, then for every $\boldsymbol{Set}$-topos $\boldsymbol{E}$ the functor $\boldsymbol{Topos/Set}(\boldsymbol{E}, Sh(\boldsymbol{P}, J)) \to \boldsymbol{Mod}_{\boldsymbol{E}}(\boldsymbol{P})$ that assigns to every $f : \boldsymbol{E} \to Sh(\boldsymbol{P}, J)$ the composition $\boldsymbol{P} \xrightarrow{M_0} Sh(\boldsymbol{P}, J) \xrightarrow{f^*} \boldsymbol{E}$ is an equivalence.*
$\square$

From [18] we know that finitary coherent theories correspond to small pretoposes, so what the theorem above says is that $Sh(\boldsymbol{P}, J)$ is the classifying topos for the coherent theory $\boldsymbol{P}$ over $\boldsymbol{Set}$, that is $Sh(\boldsymbol{P}, J) = \boldsymbol{Set}[\boldsymbol{P}]$.

We will have the opportunity to use Deligne's theorem (see 7.44 in [8])

Theorem 1.4. *A coherent topos has enough points.* $\square$

As it is pointed out in [8] the proof of Deligne's theorem resembles that of Gödel-Henkin completeness theorem for finitary first-order theories. This is done in [18].

We will use the following result as well (see 7.17 in [8]). Recall that (in [8]'s notation) a *surjection* $\boldsymbol{F} \to \boldsymbol{E}$ is a geometric morphism $\boldsymbol{F} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \boldsymbol{E}$ such that f^* reflects isomorphisms (equivalently f^* is faithful, equivalently the unit for the adjunction $f^* \dashv f_*$ is mono (see 4.11 in [8])).

Lemma 1.5. *If a Grothendieck topos* $\mathbf{E}$ *has enough points then there exists a surjection* $\mathbf{Set}/I \to \mathbf{E}$ *for some* I *in* $\mathbf{Set}$. $\square$

1.2 Some properties of pretoposes

In this section we include some properties of pretoposes we will use later on. Many more properties can be found in [18]. Following the notation in [18] we call a morphism $f : A \to B$ in a category $\mathbf{C}$ *surjective* if for every commutative diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f_0"'] & & B \\ & B_0 \arrow[ur, "m"'] & \end{tikzcd}$$

with m a monomorphism, m is necessarily an isomorphism. Then an *image* of an arrow $f : A \to B$, if it exists, is a subobject $m : B_0 \rightarrowtail B$ such that there exists a surjective $g : A \to B_0$ with

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & B_0 \arrow[ur, "m"'] & \end{tikzcd}$$

commutative. In a category with pullbacks images are unique up to isomorphism. Images are called *stable* if the pullback of a surjective is a surjective.

Lemma 1.6. *Let* $C_1, .., C_n$ *be objects in a category* $\mathbf{C}$ *with finite limits and finite coproducts. The following condition is equivalent to* $\coprod_{k=1}^n C_k$ *being stable.*

For every diagram $\coprod_{k=1}^n C_k \xrightarrow{\langle f_i \rangle} D \xleftarrow{g} A$ *the square*

$$\begin{tikzcd} \coprod_{k=1}^n P_k \arrow[r, "\coprod \pi_{k2}"] \arrow[d, "\langle \pi_{k1} \rangle"'] & \coprod_{k=1}^n C_k \arrow[d, "\langle f_k \rangle"] \\ A \arrow[r, "g"'] & D \end{tikzcd}$$

is a pullback, if for every k the square

$$\begin{tikzcd} P_k \arrow[r, "\pi_{k2}"] \arrow[d, "\pi_{k1}"'] & C_k \arrow[d, "f_k"] \\ A \arrow[r, "g"'] & D \end{tikzcd}$$

is a pullback. □

Now, fix a pretopos $\boldsymbol{P}$ for the rest of this section. We have (see 3.3.9 in [18])

Lemma 1.7. *$\boldsymbol{P}$ has stable images.* □

(see 3.3.10 in [18])

Lemma 1.8. *$\boldsymbol{P}$ has stable finite sups.* □

(see 3.3.5 in [18])

Lemma 1.9. *Given objects $P_1, .., P_n$ in $\boldsymbol{P}$ we have that for every k the k-th injection $i_k : P_i \to \coprod_{i=1}^n P_i$ is a monomorphism.* □

As a matter of fact it can be shown that a category with finite limits, stable finite sups, stable images, stable quotients of equivalence relations and stable finite disjoint sums is a pretopos. This is the definition of pretopos given in [18]. From there it follows that the definition adopted here and the one given in [1] are equivalent except for the smallness condition (see the discusion after definition 3.4.3 in [18]).

Suppose now we have a finite family $\{Q_k \underset{g_k}{\overset{f_k}{\rightrightarrows}} R\}_{k=1}^n$ in $\boldsymbol{P}$. Consider the pullback diagrams

$$\begin{tikzcd} P_{jk} \arrow[r, "q_{jk}"] \arrow[d, "p_{jk}"'] & Q_k \arrow[d, "g_k"] \\ Q_j \arrow[r, "f_j"'] & R \end{tikzcd}$$

Lemma 1.10. *With the above notations the square*

$$\begin{tikzcd} \coprod_{(j,k)} P_{jk} \arrow[r, "\langle i_j q_{jk} \rangle"] \arrow[d, "\langle i_k p_{jk} \rangle"'] & \coprod_j Q_j \arrow[d, "\langle g_j \rangle"] \\ \coprod_j Q_j \arrow[r, "\langle f_j \rangle"'] & R \end{tikzcd}$$

is a pullback.

Proof. We do it for n=2. Since finite coproducts are stable, it follows from Lemma 1.6 that for any $a : A \to P$ the following square is a pullback

$$\begin{tikzcd}
(A \times_P Q_1) \coprod (A \times_P Q_2) \arrow[r, "\langle i_1\pi_2, i_2\pi_2 \rangle"] \arrow[d, "\langle \pi_1, \pi_1 \rangle"'] & Q_1 \coprod Q_2 \arrow[d, "\langle g_1, g_2 \rangle"] \\
A \arrow[r, "a"'] & P
\end{tikzcd}$$

where $A \times_P Q_1$ is the pullback of g_1 along a and $A \times_P Q_2$ is the pullback of g_2 along a. For $a = \langle f_1, f_2 \rangle : Q_1 \coprod Q_2 \to P$ we can substitute $A \times_P Q_1$ with $P_{11} \coprod P_{21}$ and $A \times_P Q_2$ with $P_{12} \coprod P_{22}$. $\square$

Suppose now that for every $k = 1, .., n$ we have a pullback diagram

$$\begin{tikzcd}
P_k \arrow[r, "r_k"] \arrow[d, "q_k"'] & R_k \arrow[d, "g_k"] \\
Q_k \arrow[r, "f_k"'] & S_k
\end{tikzcd}$$

Lemma 1.11. *With the above notation the square*

$$\begin{tikzcd}
\coprod_k P_k \arrow[r, "\coprod_k r_k"] \arrow[d, "\coprod_k q_k"'] & \coprod_k R_k \arrow[d, "\coprod_k g_k"] \\
\coprod_k Q_k \arrow[r, "\coprod_k f_k"'] & \coprod_k S_k
\end{tikzcd}$$

is a pullback (i.e. $\coprod_k$ *preserves pullback).*

Proof. In view of Lemma 1.10 it is enough to show that for all k we have $P_k \simeq Q_k \times_{\coprod_k S_k} R_k$ and that for $j \neq k$ we have $Q_j \times_{\coprod_k S_k} R_k \simeq 0$. For the second one notice first that $S_j \times_{\coprod_k S_k} S_k \simeq 0$ since finite coproducts are disjoint, second that we can induce a map from $Q_j \times_{\coprod_k S_k} R_k$ to $S_j \times_{\coprod_k S_k} S_k$ and finally that the initial object is strict. For the first consider the diagram

$$\begin{tikzcd}
P_k \arrow[r, "r_k"] \arrow[d, "q_k"'] & R_k \arrow[r, "1_{R_k}"] \arrow[d, "g_k"] & R_k \arrow[d, "g_k"] \\
Q_k \arrow[r, "f_k"'] \arrow[d, "1_{Q_k}"'] & S_k \arrow[r, "1_{S_k}"'] \arrow[d, "1_{S_k}"] & S_k \arrow[d, "i_k"] \\
Q_k \arrow[r, "f_k"'] & S_k \arrow[r, "i_k"'] & \coprod_k S_k
\end{tikzcd}$$

Since by Lemma 1.9 the injections are mono we have that the bottom right square above is a pullback, the other three squares are also pullbacks so the exterior one is a pullback. □

Suppose we have a pair of arrows $Q \overset{f}{\underset{g}{\rightrightarrows}} P$ in $\boldsymbol{P}$. Consider the image of $\langle f, g\rangle$

$$\begin{tikzcd}
Q \arrow[rr, "\langle f,g\rangle"] \arrow[dr, "e"'] & & P \times P \\
& R \arrow[ur, "\langle r_1, r_2\rangle"'] &
\end{tikzcd}$$

We say that $\langle r_1, r_2\rangle$ is the relation *generated* by $\langle f, g\rangle$.

Lemma 1.12. *Given $Q \overset{f}{\underset{g}{\rightrightarrows}} P$ in $\boldsymbol{P}$, if there exists an arrow $\rho : P \to Q$ such that $f\rho = 1_P$ and $g\rho = 1_P$ then the relation $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} P$ generated by $\langle f, g\rangle$ is reflexive.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
& & P \\
R \arrow[urr, "r_1"] \arrow[drr, "r_2"'] & Q \arrow[l, "e"'] \arrow[ur, "f"] \arrow[dr, "g"'] & P \arrow[l, "\rho"'] \arrow[u, "1_P"'] \arrow[d, "1_P"] \\
& & P
\end{tikzcd}$$

□

Lemma 1.13. *Given $Q \overset{f}{\underset{g}{\rightrightarrows}} P$ in $\boldsymbol{P}$, if there exists an arrow $\sigma : Q \to Q$ such that* *the diagram*

$$\begin{tikzcd} & Q \arrow[dl, "f"'] \arrow[d, "\sigma"] \arrow[dr, "g"] & \\ P & Q \arrow[l, "g"] \arrow[r, "f"'] & P \end{tikzcd}$$

commutes, then the relation $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} P$ *generated by* $\langle f, g\rangle$ *is symmetric.*

Proof. By hypotheses the diagram

$$\begin{tikzcd} Q \arrow[r, "\langle f, g\rangle"] \arrow[d, "\sigma"'] & P \times P \arrow[d, "\langle \pi_2, \pi_1\rangle"] \\ Q \arrow[r, "\langle f, g\rangle"'] & P \times P \end{tikzcd}$$

commutes. Taking the image of $\langle f, g\rangle$ twice we get

$$\begin{tikzcd} Q \arrow[rr, "\langle f, g\rangle"] \arrow[dr] \arrow[dd, "\sigma"'] & & P \times P \arrow[dd, "\langle \pi_2, \pi_1\rangle"] \\ & R \arrow[ur] \arrow[d, "\hat{\sigma}"'] & \\ & R \arrow[dr] & \\ Q \arrow[rr, "\langle f, g\rangle"'] \arrow[ur] & & P \times P \end{tikzcd}$$

So there exists a unique $\hat{\sigma}$ as shown above such that the resulting diagram commutes.

□

Now we have a condition that is enough for transitivity. Given $Q \overset{f}{\underset{g}{\rightrightarrows}} P$ as before, and the generated relation $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} P$ consider the following diagram

$$\begin{tikzcd} T \arrow[rr, "p_1"] \arrow[dr, dotted, "h"'] \arrow[dd, "p_2"'] & & Q \arrow[d, "e"] \\ & S \arrow[r, "s_1"] \arrow[d, "s_2"'] & R \arrow[d, "r_2"] \\ Q \arrow[r, "e"'] & R \arrow[r, "r_1"'] & P \end{tikzcd}$$

where both squares are pullbacks. By the pullback property we can induce h above such that the resulting diagram is also commutative. Since surjections are stable and e is a surjection it is easy to see that h is also a surjection.

Lemma 1.14. *With the above notation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} P$ is transitive if there exists an arrow $t : T \to Q$ such that the diagram*

$$\begin{tikzcd} Q \arrow[d, "g"'] & T \arrow[l, "p_2"'] \arrow[r, "p_1"] \arrow[d, "t"] & Q \arrow[d, "f"] \\ P & Q \arrow[l, "g"] \arrow[r, "f"'] & P \end{tikzcd}$$

commutes.

Proof. First we show that the arrow $S \xrightarrow{\langle r_1s_1, r_2s_1, r_2s_2\rangle} P \times P \times P$ is a monomorphism. Suppose $A \underset{b}{\overset{a}{\rightrightarrows}} S$ are such that

$$A \underset{b}{\overset{a}{\rightrightarrows}} S \xrightarrow{\langle r_1s_1, r_2s_1, r_2s_2\rangle} P \times P \times P$$

commutes. Then clearly $\langle r_1, r_2\rangle s_1 a = \langle r_1, r_2\rangle s_1 b$ and $\langle r_1, r_2\rangle s_2 a = \langle r_1, r_2\rangle s_2 b$. Since $\langle r_1, r_2\rangle$ is mono we have $s_1 a = s_1 b$ and $s_2 a = s_2 b$. Since S is the pullback of r_1 and r_2 we have $a = b$. Since h is surjective we have a surjection-mono factorization

$$\begin{tikzcd} T \arrow[rr, "\langle fp_1, gp_1, gp_2\rangle"] \arrow[dr, "h"'] & & P \times P \times P \\ & S \arrow[ur, "\langle r_1s_1, r_2s_1, r_2s_2\rangle"'] & \end{tikzcd}$$

and using the properties we are assuming for t the diagram

$$\begin{tikzcd} T \arrow[r, "\langle fp_1, gp_1, gp_2\rangle"] \arrow[d, "t"'] & P \times P \times P \arrow[d, "\langle \pi_1, \pi_3\rangle"] \\ Q \arrow[r, "\langle f, g\rangle"'] & P \times P \end{tikzcd} \tag{1.1}$$

clearly commutes. Consider the following surjective-mono factorizations

$$\begin{tikzcd} S \arrow[r, tail, "\langle r_1s_1, r_2s_1, r_2s_2\rangle"] \arrow[drr, "u"'] & P \times P \times P \arrow[rr, "\langle \pi_1, \pi_3\rangle"] & & P \times P \\ & & U \arrow[ur, "u'"'] & \end{tikzcd}$$

and

$$\begin{tikzcd}
T \arrow[r, tail, "t"] \arrow[dr, "v"'] & Q \arrow[r, "e"] & R \\
& V \arrow[ur, tail, "v'"'] &
\end{tikzcd}$$

Then by the commutativity of 1.1 we have that

$$\begin{tikzcd}
T \arrow[r, two heads, "h"] \arrow[d, "v"'] & S \arrow[r, two heads, "u"] & U \arrow[d, "u'"] \arrow[dll, dotted, "\ell"'] \\
V \arrow[r, tail, "v'"'] & R \arrow[r, tail, "\langle r_1, r_2\rangle"'] & P \times P
\end{tikzcd}$$

also commutes. Notice that both compositions are surjective-mono, so we can induce ℓ as shown such that both resulting traingles commute. Define $\hat{t} : S \to R$ as the composition $S \xrightarrow{u} U \xrightarrow{\ell} V \xrightarrow{v'} R$. Now it is easy to see that

$$\begin{tikzcd}
S \arrow[d, "\hat{t}"'] \arrow[dr, "\langle r_1 s_1, r_2 s_2\rangle"] & \\
R \arrow[r, "\langle r_1, r_2\rangle"'] & P \times P
\end{tikzcd}$$

commutes. This is enough for $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} P$ to be transitive (see exercise (TRAN) in [2]).

□

1.3 Conceptual completeness

In [18] from any given finitary coherent theory they construct a pretopos that has the "same" category of models. This is done in two steps, first a logical category is constructed, a very detailed construction of it is given in [6]. The construction of a pretopos from a logical category is the second step.

The advantage of using pretoposes instead of logical categories is the following two theorems from [18], but first we need a definition (also from [18])

Definition 1.2. Given an elementary functor $F : \boldsymbol{P} \to \boldsymbol{Q}$ between pretoposes we say that

1. The functor F is subobject full iff for every P in $\boldsymbol{P}$, F induces an epimorphism $Sub(P) \to Sub(FP)$

2. The functor F is conservative iff for P in $\boldsymbol{P}$, F induces a monomorphism $Sub(P) \to Sub(FP)$

3. An object Q in $\boldsymbol{Q}$ has a finite cover via F if there exists a finite family

$$\{Q \xleftarrow{f_i} Q_i \rightarrowtail FP_i\}_{i=1}^{n}$$

such that the family $\{Q_i \xrightarrow{f_i} Q\}_{i=1}^{n}$ is epimorphic.

Observe that F being conservative is equivalent in this context to F reflecting isomorphisms.

We have (see 7.1.7 in [18])

Lemma 1.15. *If $\boldsymbol{P}$ is a pretopos then an elementary functor $F : \boldsymbol{P} \to \boldsymbol{Q}$ between pretoposes is an equivalence if and only if it satisfies the following three conditions*

1. F is subobject full.

2. F is conservative.

3. Every object of $\boldsymbol{Q}$ has a finite cover via F. □

And (see 7.1.8 in [18])

Theorem 1.16. *If $F : \boldsymbol{P} \to \boldsymbol{Q}$ is an elementary functor between small pretoposes such that $\_ \circ F : \boldsymbol{Mod}(\boldsymbol{Q}) \to \boldsymbol{Mod}(\boldsymbol{P})$ is an equivalence then F is an equivalence.*
□

Theorem 1.16 is called *conceptual completeness.* The proof in [18], besides involving lemma 1.15, involves soundness and completeness theorems and Łos-Tarski's theorem on sentences preserved by structures.

1.4 Łos' Theorem

A very important example for us of an elementary functor is given by Łos' theorem. Let $(I, \mathcal{G})$ be an ultrafilter, then we have the ultraproduct functor $\varinjlim_{J \in \mathcal{G}} \prod_{j \in J}(\_) : \boldsymbol{Set}^I \to \boldsymbol{Set}$. We also denote this functor by $\prod_I(\_)/\mathcal{G}$ or simply by $\prod_{\mathcal{G}}$. This version of Łos' Theorem comes from [15]

Theorem 1.17. *(Los' Theorem) The functor* $\varinjlim_{J\in\mathcal{G}} \prod_{j\in J}(\_) : \boldsymbol{Set}^I \to \boldsymbol{Set}$ *is elementary.*

Proof. (sketch) The proof is not hard but deserves some lines. $\prod_{\mathcal{G}}$ preserves finite limits since for every $J \subset I$ the functor $\prod_{j\in J} : \boldsymbol{Set}^J \longrightarrow \boldsymbol{Set}$ preserves limits and the colimit over elements of $\mathcal{G}$ is filtered. Since epimorphisms in $\boldsymbol{Set}^I$ are split, we have that $\prod_{\mathcal{G}}$ preserves epimorphisms. Clearly $\prod_{\mathcal{G}}$ preserves 0. Finally, given $\langle A_i\rangle, \langle B_i\rangle$ in $\boldsymbol{Set}^I$, we use the fact that $\mathcal{G}$ is an ultrafilter to show that the induced map $\prod_I A_i/\mathcal{G} + \prod_I B_i/\mathcal{G} \longrightarrow \prod_I (A_i + B_i)/\mathcal{G}$ is onto. □

1.5 Slice pretoposes

Let $\boldsymbol{P}$ be a pretopos and P an object of $\boldsymbol{P}$. We have

Lemma 1.18. *The slice category* $\boldsymbol{P}/P$ *is a pretopos*

Proof. Since $\boldsymbol{P}$ is left exact then $\boldsymbol{P}/P$ is left exact. If 0 is the initial object in $\boldsymbol{P}$ then $0 \to P$ is a strict initial object in $\boldsymbol{P}/P$. The coproduct of $Q \xrightarrow{q} P$ and $R \xrightarrow{r} P$ is $Q \coprod R \xrightarrow{[q,r]} P$ and is easily shown to be disjoint and stable. If a pair of arrows $q \underset{k}{\overset{h}{\rightrightarrows}} r$ in $\boldsymbol{P}/P$ with $Q \xrightarrow{q} P$ and $R \xrightarrow{r} P$ is an equivalence relation then the corresponding $Q \underset{k}{\overset{h}{\rightrightarrows}} R$ is an equivalence relation in $\boldsymbol{P}$. Consider its quotient $R \xrightarrow{\ell} S$ in $\boldsymbol{P}$. Using the universal property of the quotient we induce a map $S \xrightarrow{s} P$ such that $r \xrightarrow{\ell} s$ is a morphism in $\boldsymbol{P}/P$. This last arrow is the quotient in $\boldsymbol{P}/P$. □

Then we have the forgetful functor $U : \boldsymbol{P}/P \to \boldsymbol{P}$ that has a right adjoint $\Delta_P : \boldsymbol{P} \longrightarrow \boldsymbol{P}/P$. Given $f : Q \to R$ in $\boldsymbol{P}$ we have that $\Delta_P(Q) = \pi_P : Q \times P \to P$ and $\Delta_P(f) = f \times P$. We are ready for

Proposition 1.19. *The functor* $\Delta_P : \boldsymbol{P} \longrightarrow \boldsymbol{P}/P$ *is elementary.*

Proof. Δ_P clearly preserves finite limits since it has a left adjoint. $\Delta_P(0) = \pi_P : 0 \times P \to P$ but $0 \times P \simeq 0$ due to the fact that 0 is strict in $\boldsymbol{P}$. Since binary coproducts are stable and for every Q, R in $\boldsymbol{P}$ we have that both squares in the diagram

$$\begin{tikzcd}
Q \times P \arrow[r, "i_Q \times P"] \arrow[d, "\pi_Q"'] & (Q \coprod R) \times P \arrow[d, "\pi_1"] & R \times P \arrow[l, "i_R \times P"'] \arrow[d, "\pi_R"] \\
Q \arrow[r, "i_Q"'] & Q \coprod R & R \arrow[l, "i_R"]
\end{tikzcd}$$

are pullbacks, we have that $(Q \coprod R) \times P \simeq (Q \times P) \coprod (R \times P)$. Then Δ_P preserves binary coproducts. The proof for preserving quotients of equivalence relations is left to the reader. $\square$

For any pretopos $\boldsymbol{A}$ we can induce the functor

$$- \circ \Delta_P : \boldsymbol{Mod_A}(\boldsymbol{P}/P) \to \boldsymbol{Mod_A}(\boldsymbol{P}).$$

What we want to do now is to give an equivalent description of the category $\boldsymbol{Mod_A}(\boldsymbol{P}/P)$ in terms of the category $\boldsymbol{Mod_A}(\boldsymbol{P})$.

Define the category $El_{\boldsymbol{A}}(ev_P)$ as follows. The objects of $El_{\boldsymbol{A}}(ev_P)$ are pairs (M, a), where $M \in \boldsymbol{Mod_A}(\boldsymbol{P})$ and a is a global element of MP, that is, $a : 1 \to MP$ in $\boldsymbol{A}$. An arrow $h : (M, a) \to (N, b)$ in $El(ev_P)$ is an arrow $h : M \to N$ in $\boldsymbol{Mod_A}(\boldsymbol{P})$ such that the diagram

$$\begin{tikzcd}
1 \arrow[rr, "a"] \arrow[dr, "b"'] & & MP \arrow[dl, "hP"] \\
& NP &
\end{tikzcd}$$

commutes. As usual, when $\boldsymbol{A} = \boldsymbol{Set}$ we drop the subscript.

Theorem 1.20. *If $\boldsymbol{A}$ is a pretopos then the categories $El_{\boldsymbol{A}}(ev_P)$ and $\boldsymbol{Mod_A}(\boldsymbol{P}/P)$ are equivalent.*

Proof.- We define a functor $\Theta : El_{\boldsymbol{A}}(ev_P) \to \boldsymbol{Mod_A}(\boldsymbol{P}/P)$ as follows. Given (M, a) in $El_{\boldsymbol{A}}(ev_P)$ define $\Theta(M, a) : \boldsymbol{P}/P \to \boldsymbol{A}$ such that $\Theta(M, a)(Q \xrightarrow{q} P)$ is the pullback

$$\begin{tikzcd}
\Theta(M,a)(Q \xrightarrow{q} P) \arrow[r] \arrow[d] & MQ \arrow[d, "Mq"] \\
1 \arrow[r, "a"'] & MP,
\end{tikzcd}$$

and if

$$\begin{tikzcd} Q \arrow[rr, "f"] \arrow[dr, "q"'] & & R \arrow[dl, "r"] \\ & P & \end{tikzcd}$$

is a morphism in $\boldsymbol{P}/P$, we define $\Theta(M,a)(f) : \Theta(M,a)(Q \xrightarrow{q} P) \to \Theta(M,a)(Q \xrightarrow{r} P)$ as the unique morphism that makes the diagram

$$\begin{tikzcd} \Theta(M,a)(q) \arrow[rrr] \arrow[dr, "\Theta(M,a)(f)"] \arrow[ddr] & & & MQ \arrow[dl] \arrow[ddl, "Mq"] \\ & \Theta(M,a)(r) \arrow[r] \arrow[d] & MR \arrow[d, "Mr"'] & \\ & 1 \arrow[r, "a"'] & MP & \end{tikzcd}$$

commute. $\Theta(M,a)$ turns out to be an elementary functor from $\boldsymbol{P}/P$ to $\boldsymbol{A}$. Now, if $h : (M,a) \to (M',b)$ is in $El_{\boldsymbol{A}}(ev_P)$, then define $\Theta(h) : \Theta(M,a) \to \Theta(M',b)$ such that for every $Q \xrightarrow{q} P$ in $\boldsymbol{P}/P$, $\Theta(h)(q)$ is the unique morphism that makes the diagram

$$\begin{tikzcd} \Theta(M',b)(q) \arrow[rrr] \arrow[ddd] & & & M'Q \arrow[ddd, "M'q"] \\ & \Theta(M,a)(q) \arrow[ul, "\Theta(h)(q)"'] \arrow[r] \arrow[d] & MQ \arrow[ur, "hQ"'] \arrow[d, "Mq"] & \\ & 1 \arrow[r, "a"'] \arrow[dl] & MP \arrow[dr, "hP"] & \\ 1 \arrow[rrr, "b"'] & & & M'P \end{tikzcd}$$

commute. We define now a functor in the other direction. Define $\Xi : \boldsymbol{Mod_A}(\boldsymbol{P}/P) \to El_{\boldsymbol{A}}(ev_P)$ as follows. Given a model N in $\boldsymbol{Mod_A}(\boldsymbol{P}/P)$, when we apply N to

$$\begin{tikzcd} P \arrow[rr, "\delta"] \arrow[dr, "1_P"'] & & P \times P \arrow[dl, "\pi_2"] \\ & P & \end{tikzcd}$$

where δ is the diagonal map, we obtain a morphism $N\delta : 1 \to N(\Delta_P(P))$. We define $\Xi(N) = (N \circ \Delta_P, N\delta)$. If $k : N \to N'$ is a morphism in $\boldsymbol{Mod_A}(\boldsymbol{P}/P)$ then it is clear that the diagram

$$\begin{tikzcd} 1 \arrow[r, "N\delta"] \arrow[dr, "N'\delta"'] & N(\Delta_P(P)) \arrow[d, "k\Delta_P(P)"] \\ & N'(\Delta_P(P)) \end{tikzcd}$$

commutes. Define $\Xi(k) = k\Delta_P : (N \circ \Delta_P, N\delta) \to (N' \circ \Delta_P, N'\delta)$. It is not hard to prove that Ξ is a quasi-inverse for Θ. $\square$

It is easy to see that the forgetful functor $El_{\boldsymbol{A}}(ev_P) \to \boldsymbol{Mod_A}(\boldsymbol{P})$, $(M, a) \mapsto M$ is isomorphic to the composition $El_{\boldsymbol{A}}(ev_P) \xrightarrow{\Theta} \boldsymbol{Mod_A}(\boldsymbol{P}/P) \xrightarrow{-\circ\Delta_P} \boldsymbol{Mod_A}(\boldsymbol{P})$.

We use this description to give a categorical proof, instead of the usual model theoretic argument, of the following theorem from [15] we will need later. First a little notation. Given an ultrafilter $(I, \mathcal{G})$, we have the ultraproduct functor

$$\prod_{\mathcal{G}} = \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} (\_) : \boldsymbol{Mod}(\boldsymbol{P})^I \to \boldsymbol{Mod}(\boldsymbol{P}).$$

If we have a family of models $\langle M_i \rangle_I$ we denote $\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} (\langle M_i \rangle_I)$ by $\prod_I M_i/\mathcal{G}$. When we apply this functor to the constant I-family $\langle M \rangle_I$ we denote the result by $M^{\mathcal{G}}$. We denote by $\delta : M \to M^{\mathcal{G}}$ the usual diagonal morphism. If we have a monomorphism $Q \rightarrowtail P$ in $\boldsymbol{P}$ and a model M in $\boldsymbol{Mod}(\boldsymbol{P})$, we have that $MQ \rightarrowtail MP$. We may assume that this mono is actual containment of sets. If we have a homomorphism $h : N \to M^{\mathcal{G}}$ and elements $a \in MP, b \in NP$ for some P in $\boldsymbol{P}$ such that $hP(b) = \delta P(a)$, then it is not hard to see that for every $Q \rightarrowtail P$ in $\boldsymbol{P}$, $b \in NQ$ implies $a \in MQ$. The converse also holds.

Theorem 1.21. *Assume $\boldsymbol{P}$ is small. Let $(M, a), (N, b) \in El(ev_P)$, suppose that for every monomorphism $Q \rightarrowtail P$ we have that $b \in NQ$ implies $a \in MQ$, then there exist an ultrafilter $(I, \mathcal{G})$ and a homomorphism $h : N \longrightarrow M^{\mathcal{G}}$ such that $hP(b) = \delta P(a)$.*

We will prove the case $P = 1$ first

Lemma 1.22. *Let M, N in $\boldsymbol{Mod}(\boldsymbol{P})$, suppose that for every monomorphism $Q \rightarrowtail 1$, $NQ = 1$ implies $MQ = 1$, then there exist an ultrafilter $(S, \mathcal{G})$ and a homomorphism $N \longrightarrow M^{\mathcal{G}}$.*

Proof. Notice first that the condition of the lemma is equivalent to saying that for every $P \in \boldsymbol{P}$ such that $NP \neq \emptyset$ we have that $MP \neq \emptyset$. To see this, consider the image of $P \longrightarrow 1$. Since N preserves images, if $NP \neq \emptyset$, then N of the image must be 1. Then M of the image must also be 1, therefore $MP \neq \emptyset$. The converse is clear. Define S to be the set of finitely generated subcategories of $El(N)$. If $\boldsymbol{I} \in S$, there exists a diagram $\Gamma_{\boldsymbol{I}} : \boldsymbol{I} \longrightarrow El(M)$ such that the diagram

$$\begin{tikzcd} \boldsymbol{I} \arrow[r, "i"] \arrow[d, "\Gamma_{\boldsymbol{I}}"'] & El(N) \arrow[d] \\ El(M) \arrow[r] & \boldsymbol{P} \end{tikzcd}$$

commutes, where the functors to $\boldsymbol{P}$ are forgetful functors. To show this, consider the diagram $\boldsymbol{I} \xrightarrow{i} El(N) \longrightarrow \boldsymbol{P}$. Since $\boldsymbol{P}$ has finite limits and $\boldsymbol{I}$ is finitely generated we have that the limit $\varprojlim_{(b \in NP) \in \boldsymbol{I}} P$ of the diagram exists in $\boldsymbol{P}$. It is clear that $N(\varprojlim_{(b \in NP) \in \boldsymbol{I}} P) \cong \varprojlim_{(b \in NP) \in \boldsymbol{I}} NP$. We have $\langle b \rangle_{(b \in NP) \in \boldsymbol{I}} \in \varprojlim_{(b \in NP) \in \boldsymbol{I}} NP$. Then $\varprojlim_{(b \in NP) \in \boldsymbol{I}} NP \neq \emptyset$. It follows that $\varprojlim_{(b \in NP) \in \boldsymbol{I}} MP \neq \emptyset$. But an element in $\varprojlim_{(b \in NP) \in \boldsymbol{I}} MP$ determines a $\Gamma_{\boldsymbol{I}} : \boldsymbol{I} \longrightarrow El(M)$ such that the square above commutes. For every $\boldsymbol{I} \in S$ choose a $\Gamma_{\boldsymbol{I}}$. Given $\boldsymbol{I} \in S$, let $\uparrow(\boldsymbol{I}) = \{\boldsymbol{K} \in S | \boldsymbol{I} \subset \boldsymbol{K}\}$. It is clear that $\uparrow(\boldsymbol{I}) \neq \emptyset$. Given $\boldsymbol{I}$ and $\boldsymbol{I}'$ in S, Let $\boldsymbol{J}$ be the subcategory of $El(N)$ generated by $\boldsymbol{I} \cup \boldsymbol{I}'$. Clearly $\boldsymbol{J} \in S$, and $\uparrow(\boldsymbol{I}) \cap \uparrow(\boldsymbol{I}') = \uparrow(\boldsymbol{J})$. Let $\mathcal{G}$ be an ultrafilter on S such that for every $\boldsymbol{I} \in S$ we have that $\uparrow(\boldsymbol{I}) \in \mathcal{G}$. Consider the ultrapower $M^{\mathcal{G}}$, and define $h : N \longrightarrow M^{\mathcal{G}}$ as follows. Given $b \in NP$ consider the subcategory of $El(N)$ that consists of one object, $(b \in NP)$, and its identity arrow. Let $hP(b) = \langle \Gamma_{\boldsymbol{I}}(b \in NP) \rangle_{\boldsymbol{I} \in \uparrow(b \in NP)}$. So, we have a function $hP : NP \longrightarrow M^{\mathcal{G}}P$. We have to show that h is natural. Let $f : P \longrightarrow P'$ in $\boldsymbol{P}$, consider the diagram

$$\begin{tikzcd} NP \arrow[r, "hP"] \arrow[d, "Nf"'] & M^{\mathcal{G}}P \arrow[d, "M^{\mathcal{G}}f"] \\ NP' \arrow[r, "hP'"'] & M^{\mathcal{G}} \end{tikzcd}$$

Let $b \in NP$, and let $\boldsymbol{I}$ be the subcategory of $El(N)$ generated by $(b \in NP) \xrightarrow{f}$

$(Nf(b) \in NP')$. For every $\boldsymbol{J} \in S_{\boldsymbol{I}}$ we have that $Mf(\Gamma_{\boldsymbol{I}}(b \in NP)) = \Gamma_{\boldsymbol{I}}(Nf(b) \in NP')$. Therefore the previous square commutes. □

The proof of the next lemma is easy

Lemma 1.23. *Let $(M, a), (N, b) \in El(ev_P)$, the following two statements are equivalent;*

For every monomorphism $Q \rightarrowtail P$, $b \in NQ$ implies $a \in MQ$

For every monomorphism $r \rightarrowtail 1$ in $\boldsymbol{P}/P$, $\Theta(N, b)(r) = 1$ implies $\Theta(M, b)(r) = 1$

□

Proof of theorem 1.21.- Suppose that for every monomorphism $Q \rightarrowtail P$ we have that $b \in NQ$ implies $a \in MP$, then, by lemma 1.22 there exist a filter $(S, \mathcal{G})$ and a homomorphism $k : \Theta(N, b) \longrightarrow \Theta(M, a)^{\mathcal{G}}$. This corresponds to a homomorphism $h : N \longrightarrow M^{\mathcal{G}}$ such that $hP(b) = \delta P(a)$. □

1.6 Left exact categories and pretoposes

It is shown in [18] that given a small site $(\boldsymbol{C}, J)$ with $\boldsymbol{C}$ a left exact category and J generated by a pretopology (in the sense of [8]) all of whose covering families are finite, a small pretopos $F(\boldsymbol{C}, J)$ can be constructed such that the category $\boldsymbol{Mod}(F(\boldsymbol{C}, J))$ is equivalent to $Sh(\boldsymbol{C}, J)$. This is done by producing first a theory $\boldsymbol{T}_{(\boldsymbol{C},J)}$ such that for any logical category $\boldsymbol{R}$, $\boldsymbol{R}$-models of $(\boldsymbol{C}, J)$ are "the same thing" as $\boldsymbol{R}$ models of $\boldsymbol{T}_{(\boldsymbol{C},J)}$ (see 6.1.1 in [18]). From $\boldsymbol{T}_{(\boldsymbol{C},J)}$ a logical category $R(\boldsymbol{C}, J)$ is constructed together with a canonical model $M_0 : \boldsymbol{T}_{(\boldsymbol{C},J)} \to R(\boldsymbol{C}, J)$ with the universal property that for every logical category $\boldsymbol{R}$, $\boldsymbol{R}$ models of $\boldsymbol{T}_{(\boldsymbol{C},J)}$ are "the same thing" as logical functors from $R(\boldsymbol{C}, J)$ to $\boldsymbol{R}$, the passage given by M_0. Finally $R(\boldsymbol{C}, J)$ is completed to a pretopos $F(\boldsymbol{C}, J)$ and a logical functor $N_0 : R(\boldsymbol{C}, J) \to F(\boldsymbol{C}, J)$ with the universal property that for every pretopos $\boldsymbol{P}$, logical functors from $R(\boldsymbol{C}, J)$ to $\boldsymbol{P}$ are in correspondence with elementary functors from $F(\boldsymbol{C}, J)$ to $\boldsymbol{P}$. In particular, when J is generated by the pretopology whose covering families are singletons containing isomorphisms a $\boldsymbol{P}$ model of $(\boldsymbol{C}, J)$ is simply a left exact functor from $\boldsymbol{C}$ to $\boldsymbol{P}$. Then the construction described above gives a left exact functor $F_0 : \boldsymbol{C} \to F(\boldsymbol{C}, J)$ with the universal property that composition with F_0 induces an equivalence from $\boldsymbol{Mod}_{\boldsymbol{P}}(F(\boldsymbol{C}, J))$ to $\boldsymbol{Lex}(\boldsymbol{C}, \boldsymbol{P})$ for any pretopos $\boldsymbol{P}$. We have a forgetful functor

$U : \boldsymbol{Pretop} \to \boldsymbol{Lex}$. The discussion above gives a small pretopos $F(\boldsymbol{C})$ for every left exact category $\boldsymbol{C}$ together with a universal functor $F_0 : \boldsymbol{C} \to F(\boldsymbol{C})$. This clearly produces a left adjoint for U. $F(\boldsymbol{C})$ turns out to be the category $(\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh}$ (see 9.2.5 in [18]). What we do in this section is to give a combinatorial description of $F(\boldsymbol{C})$ using only $\boldsymbol{C}$.

1.6.1 Coherent objects of $\boldsymbol{Set}^{\boldsymbol{C}^{op}}$

Start then with a small left exact category $\boldsymbol{C}$.

Lemma 1.24. *A functor $F : \boldsymbol{C}^{op} \to \boldsymbol{Set}$ is a compact object in $\boldsymbol{Set}^{\boldsymbol{C}^{op}}$ if and only if it is finitely generated (that is, there exist objects $C_1, .., C_n$ in $\boldsymbol{C}$ and an epimorphism $\coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k) \twoheadrightarrow F$)*

Proof. Suppose F is compact. For every $x \in FC$ consider $\tau_{(x\in FC)} : \boldsymbol{C}(\_, C) \to F$ such that $\tau_{(x\in FC)}C(1_C) = x$. Then the family $\{\boldsymbol{C}(\_, C) \xrightarrow{\tau_{(x\in FC)}} F\}_{x\in FC}$ is an epimorphic family. Since F is compact there exist $x_1 \in FC_1, .., x_n \in FC_n$ such that $\{\boldsymbol{C}(\_, C_k) \xrightarrow{\tau_{(x_k\in FC_k)}} F\}_{k=1}^{n}$ is an epimorphic family. This clearly means that $\langle \tau_{(x_k\in FC_k)} \rangle : \coprod_k \boldsymbol{C}(\_, C_k) \twoheadrightarrow F$ is an epimorphism.

Assume now that we have an epimorphism $\langle \tau_k \rangle : \coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k) \twoheadrightarrow F$ and an epimorphic family $\{G_\alpha \xrightarrow{f_\alpha} F\}_\alpha$. Then for every $k = 1, .., n$ there exists some α_k and $x_k \in G_{\alpha_k}C_k$ such that $f_{\alpha_k}C_k(x_k) = \tau_k C_k(1_{C_k})$. It follows that the family $\{G_{\alpha_k} \xrightarrow{f_{\alpha_k}} F\}_{k=1}^{n}$ is an epimorphic family. $\square$

Proposition 1.25. *A functor $F : \boldsymbol{C}^{op} \to \boldsymbol{Set}$ is a coherent object if and only if there is a coequalizer of the form*

$$\coprod_{j=1}^{m} \boldsymbol{C}(\_, D_j) \rightrightarrows \coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k) \twoheadrightarrow F$$

in $\boldsymbol{Set}^{\boldsymbol{C}^{op}}$ such that $\coprod_{j=1}^{m} \boldsymbol{C}(\_, D_j) \rightrightarrows \coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k)$ generates an equivalence relation

Proof. Let F in $(\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh}$. By Proposition 1.24 we can find an epimorphism $\coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k) \xrightarrow{\langle \tau_k \rangle} F$. Consider its kernel pair $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} \coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k)$. Since R is compact (it is coherent by Theorem 1.1) there exists an epimorphism

$$\coprod_{j=1}^{m} C(-, D_j) \xrightarrow{\langle \beta_j \rangle} R.$$

This produces a coequalizer diagram

$$\coprod_{j=1}^{m} C(-, D_j) \rightrightarrows \coprod_{k=1}^{n} C(-, C_k) \twoheadrightarrow F.$$

with (r_1, r_2) the equivalence relation generated by the pair of arrows on the left in the diagram above.

Conversely, assume $\coprod_{j=1}^{m} C(-, D_j) \rightrightarrows \coprod_{k=1}^{n} C(-, C_k) \twoheadrightarrow F$ is a coequalizer such that the pair of arrows on the left generates an equivalence relation $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} \coprod_{k=1}^{n} C(-, C_k)$. Since $\coprod_{j=1}^{m} C(-, D_j)$ and $\coprod_{k=1}^{n} C(-, C_k)$ are coherent and images in $(\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh}$ are calculated as in $\boldsymbol{Set}^{\boldsymbol{C}^{op}}$ we conclude that R is coherent. Since (r_1, r_2) is an equivalence relation with coequalizer F it follows that F is coherent. $\square$

Remark 1.1. Without the equivalence relation condition in Proposition 1.25 we would simply have that F is finitely presentable. So being coherent is a stronger condition on a functor F that being finitely presentable.

1.6.2 Free Pretopos Generated By a Left Exact Category

Considering the previous section, the idea to construct the pretopos from $\boldsymbol{C}$ is to characterize the pairs of arrows of the form

$$\coprod_{j=1}^{m} C(-, D_j) \rightrightarrows \coprod_{k=1}^{n} C(-, C_k)$$

that generate equivalence relations (that is, that the image of

$$\coprod_{j=1}^{m} C(-, D_j) \longrightarrow (\coprod_{k=1}^{n} C(-, C_k)) \times (\coprod_{k=1}^{n} C(-, C_k))$$

is an equivalence relation).

Notice that an arrow $\coprod_{j=1}^{m} \boldsymbol{C}(\_, D_j) \xrightarrow{\gamma} \coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k)$ is a j-family of arrows

$$\{\boldsymbol{C}(\_, D_j) \xrightarrow{\gamma_j} \coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k)\}_{j=1}^{m}$$

and that this in turn corresponds to a family of arrows $\{D_j \xrightarrow{f_j} C_{k_j}\}_{j=1}^{m}$. That is $\gamma = \langle \boldsymbol{C}(\_, f_j)\rangle_j$. Or put another way, there exists a function $f : \{1, 2, .., m\} \to \{1, .., n\}$ and a family of arrows $\{f_j : D_j \to C_{f(j)}\}_{j=1}^{m}$ such that for every j the diagram

$$\begin{tikzcd}
\boldsymbol{C}(_, D_j) \arrow[r, "\boldsymbol{C}(_, f_j)"] \arrow[d, "i_j"'] & \boldsymbol{C}(_, C_{f(j)}) \arrow[d, "i_{f(j)}"] \\
\coprod_{j=1}^{m} \boldsymbol{C}(_, D_j) \arrow[r, "\gamma"'] & \coprod_{k=1}^{n} \boldsymbol{C}(_, C_k)
\end{tikzcd}$$

commutes. Let's start with two functions $\{1, .., m\} \overset{f}{\underset{g}{\rightrightarrows}} \{1, .., n\}$ and two families of arrows $\{f_j : D_j \to C_{f(j)}\}_{j=1}^{m}$ and $\{g_j : D_j \to C_{g(j)}\}_{j=1}^{m}$ in $\boldsymbol{C}$, and assume that

$$\coprod_{j=1}^{m} \boldsymbol{C}(\_, D_j) \overset{\langle i_{f(j)} \circ \boldsymbol{C}(\_, f_j)\rangle}{\underset{\langle i_{g(j)} \circ \boldsymbol{C}(\_, g_j)\rangle}{\rightrightarrows}} \coprod_{k=1}^{n} \boldsymbol{C}(\_, C_k)$$

generates a reflexive relation. Consider then the epi-mono factorization

$$\begin{tikzcd}
\coprod_{j=1}^{m} \boldsymbol{C}(_, D_j) \arrow[rr, "\langle \boldsymbol{C}(_, f_j), \boldsymbol{C}(_, g_j)\rangle"] \arrow[dr, "\beta"'] & & \coprod_{k=1}^{n} \boldsymbol{C}(_, C_k) \times \coprod_{k=1}^{n} \boldsymbol{C}(_, C_k) \\
& R \arrow[ur, "\langle r_1, r_2\rangle"'] &
\end{tikzcd}$$

We are supposing then that (r_1, r_2) is a reflexive relation. Then there exists an arrow $\tau : \coprod \boldsymbol{C}(\_, C_k) \to R$ such that the diagram

$$\begin{tikzcd}
\coprod_{k=1}^{n} \boldsymbol{C}(_, C_k) \arrow[rr, "\Delta"] \arrow[dr, "\tau"'] & & \coprod_{k=1}^{n} \boldsymbol{C}(_, C_k) \times \coprod_{k=1}^{n} \boldsymbol{C}(_, C_k) \\
& R \arrow[ur, "\langle r_1, r_2\rangle"'] &
\end{tikzcd}$$

commutes. Since β is epi we can find a function $r : \{1, .., n\} \to \{1, .., m\}$ and a family of arrows $\{C_k \xrightarrow{r_k} D_{r(k)}\}_{k=1}^{n}$ such that for every k, $\beta C_k(r_k) = \tau C_k(1_{C_k})$. This implies that for every k, $f_{r(k)}r_k = 1_{C_k} = g_{r(k)}r_k$. It follows that $fr = 1_{\{1,..,m\}} = gr$ and that the diagram

$$\begin{tikzcd}
\coprod_{k=1}^{n} \boldsymbol{C}(\text{-}, C_k) & \coprod_{k=1}^{n} \boldsymbol{C}(\text{-}, C_k) \arrow[l, "1"'] \arrow[r, "1"] \arrow[d, "\langle i_{r(k)}\boldsymbol{C}(\text{-},C_k)\rangle"'] & \coprod_{j=1}^{m} \boldsymbol{C}(\text{-}, D_j) \\
& \coprod_{k=1}^{n} \boldsymbol{C}(\text{-}, C_k) \arrow[ul, "\langle \boldsymbol{C}(\text{-}, g_j)\rangle"] \arrow[ur, "\langle \boldsymbol{C}(\text{-}, g_j)\rangle"'] &
\end{tikzcd}$$

commutes. The existence of a commutative diagram as above implies that the generated relation is reflexive. We will have to take care of symmetry and transitivity in the same way, and show that they work in any pretopos.

For the formal construction that follows we are going to use the concept of limit sketch, for which we refer the reader to [16].

Let $\mathcal{S}$ be the limit sketch $\mathcal{S} = (\boldsymbol{G}, D, L)$, where $\boldsymbol{G}$ is the graph

$$\begin{tikzcd}
2 \arrow[r, shift left=2, "p_{12}"] \arrow[r, "t"] \arrow[r, shift right=2, "p_{01}"'] & 1 \arrow[loop above, "s"] \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] & 0 \arrow[l, "r"'] ,
\end{tikzcd}$$

D consists of the following diagrams

$$\begin{tikzcd}
0 \arrow[r, "r"] \arrow[d, "1_0"'] & 1 \arrow[dl, "g"] \\
0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, "r"] \arrow[d, "1_0"'] & 1 \arrow[dl, "f"] \\
0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "f"] \arrow[d, "s"'] & 0 \\
1 \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "g"] \arrow[d, "s"'] & 0 \\
1 \arrow[ur, "f"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
2 \arrow[r, "t"] \arrow[d, "p_{01}"'] & 1 \arrow[d, "f"] \\
1 \arrow[r, "f"'] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[r, "p_{12}"] \arrow[d, "t"'] & 1 \arrow[d, "g"] \\
1 \arrow[r, "g"'] & 0
\end{tikzcd}$$

and L only has the cone

$$\begin{tikzcd}
2 \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] & 1 \arrow[d, "f"] \\
1 \arrow[r, "g"'] & 0
\end{tikzcd}$$

We are going to consider models of the sketch $\mathcal{S}$ in $\boldsymbol{Set}_0$. Given a model $\Phi : \mathcal{S} \to \boldsymbol{Set}_0$ we are thinking of $\Phi(0)$ as the set $\{1, .., n\}$ and $\Phi(1)$ as the set $\{1, .., m\}$ in the discussion above, and f, g and r as the functions with the same names as above. The introduction of the pullback $\Phi(2)$ is necessary for transitivity. The names in the graph $\boldsymbol{G}$ are not accidental, r relates to reflexivity, s to symmetry and t to transitivity. Notice that the diagrams in D that have r in them represent the condition on the indexing sets that we found necessary on the discussion above for the generated relation to be reflexive.

For every model $\Phi : \mathcal{S} \to \boldsymbol{Set}_0$ we can construct a new limit sketch $\mathcal{S}_\Phi = (\boldsymbol{G}_\Phi, D_\Phi, L_\Phi)$ as follows. The graph $\boldsymbol{G}_\Phi$ has as set of nodes the set $\Phi(0) \coprod \Phi(1) \coprod \Phi(2)$. To make the notation easier we are going to denote the elements of $\Phi(0)$ by the variable x, possibly with subindexes, the elements of $\Phi(1)$ by the variable y again with possible subindexes and the elements of $\Phi(2)$ as pairs (y_1, y_2). We have the following arrows in $\mathcal{G}_\Phi$

$y \xrightarrow{f} \Phi f(y)$ for every $y \in \Phi(1)$.

$y \xrightarrow{g} \Phi g(y)$ for every $y \in \Phi(1)$.

$x \xrightarrow{r} \Phi r(x)$ for every $x \in \Phi(0)$.

$y \xrightarrow{s} \Phi s(y)$ for every $y \in \Phi(1)$.

$(y_1, y_2) \xrightarrow{t} \Phi t(y_1, y_2)$ for every $(y_1, y_2) \in \Phi(2)$.

$(y_1, y_2) \xrightarrow{p_{01}} \Phi(y_1, y_2) = y_1$ for every $(y_1, y_2) \in \Phi(2)$.

$(y_1, y_2) \xrightarrow{p_{12}} \Phi p_{12}(y_1, y_2) = y_2$ for every $(y_1, y_2) \in \Phi(2)$.

Notice that we have given the same name to many different arrows, if $y_1 \neq y_2$ then $((y_1 \xrightarrow{f} \Phi f(y_1)) \neq (y_2 \xrightarrow{f} \Phi f(y_2))$ so it will be necessary to specify domain and codomain when confusion may arise.

D_Φ mirrors D in the following way. For every $x \in \Phi(0), y \in \Phi(1)$ and $(y_1, y_2) \in \Phi(2))$ the following diagrams are in D_Φ.

$$\begin{tikzcd} x \arrow[r, "r"] \arrow[d, "1_x"'] & \Phi r(x) \arrow[dl, "g"] \\ x & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "r"] \arrow[d, "1_x"'] & \Phi r(x) \arrow[dl, "f"] \\ x & \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "f"] \arrow[d, "s"'] & \Phi f(y) \\ \Phi s(y) \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "g"] \arrow[d, "s"'] & \Phi g(y) \\ \Phi s(y) \arrow[ur, "f"'] & \end{tikzcd}$$

$$\begin{tikzcd}
(y_1, y_2) \arrow[r, "t"] \arrow[d, "p_{01}"'] & \Phi t(y_1, y_2) \arrow[d, "f"] \\
y_1 \arrow[r, "f"'] & \Phi f(y_1)
\end{tikzcd}
\qquad
\begin{tikzcd}
(y_1, y_2) \arrow[r, "p_{12}"] \arrow[d, "t"'] & y_2 \arrow[d, "g"] \\
\Phi t(y_1, y_2) \arrow[r, "g"'] & \Phi g(y_2)
\end{tikzcd}$$

and for every $(y_1, y_2) \in \Phi(2)$, L_Φ has the cone

$$\begin{tikzcd}
(y_1, y_2) \arrow[r, "p_{12}"] \arrow[d, "p_{01}"'] & y_2 \arrow[d, "f"] \\
y_1 \arrow[r, "g"'] & \Phi f(y_1)
\end{tikzcd}$$

Given a left exact category $\boldsymbol{C}$ we are going to consider models $\Gamma : \mathcal{S}_\Phi \to \boldsymbol{C}$. We will denote $\Gamma(x)$ by Γ_x, similarly $\Gamma(y)$ by Γ_y and $\Gamma(y_1, y_2)$ by $\Gamma_{y_1 y_2}$

When we have a pretopos $\boldsymbol{P}$ instead of just a left exact category and a model $\Gamma : \mathcal{S}_\Phi \to \boldsymbol{P}$ we can induce arrows $\varphi, \psi : \coprod_{y \in \Phi(1)} \Gamma_y \to \coprod_{x \in \Phi(0)} \Gamma_x$ such that the diagrams

$$\begin{tikzcd}
\Gamma_y \arrow[r, "\Gamma f"] \arrow[d, "i_y"'] & \Gamma_{\Phi f(y)} \arrow[d, "i_{\Phi f(y)}"] \\
\coprod_{y \in \Phi(1)} \Gamma_y \arrow[r, "\varphi"'] & \coprod_{x \in \Phi(0)} \Gamma_x
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma_y \arrow[r, "\Gamma g"] \arrow[d, "i_y"'] & \Gamma_{\Phi g(y)} \arrow[d, "i_{\Phi g(y)}"] \\
\coprod_{y \in \Phi(1)} \Gamma_y \arrow[r, "\psi"'] & \coprod_{x \in \Phi(0)} \Gamma_x
\end{tikzcd} \tag{1.2}$$

commute. Then we can consider the relation generated by (φ, ψ), that is, the image

$$\begin{tikzcd}
\coprod_{y \in \Phi(1)} \Gamma_y \arrow[rr, "\langle \varphi, \psi \rangle"] \arrow[dr, "e"'] & & \coprod_{x \in \Phi(0)} \Gamma_x \times \coprod_{x \in \Phi(0)} \Gamma_x \\
& R \arrow[ur, "\langle r_1, r_2 \rangle"'] &
\end{tikzcd}$$

Proposition 1.26. *Given a pretopos $\boldsymbol{P}$, a model $\Phi : \mathcal{S} \to \boldsymbol{Set}_0$ and a model $\Gamma : \mathcal{S}_\Phi \to \boldsymbol{P}$, induce $\varphi, \psi : \coprod_{y \in \Phi(1)} \Gamma_y \to \coprod_{x \in \Phi(0)} \Gamma_x$ as above. The relation generated by (φ, ψ) is an equivalence relation.*

Proof. Induce $\rho : \coprod_{x \in \Phi(0)} \Gamma_x \to \coprod_{y \in \Phi(1)} \Gamma_y$ such that for every $x \in \Phi(0)$ the diagram

$$\begin{tikzcd}
\Gamma_x \arrow[r, "\Gamma r"] \arrow[d, "i_x"'] & \Gamma_{\Phi r(x)} \arrow[d, "i_{\Phi r(x)}"] \\
\coprod_{x \in \Phi(0)} \Gamma_x \arrow[r, "\rho"'] & \coprod_{y \in \Phi(1)} \Gamma_y
\end{tikzcd}$$

commutes. Since the diagrams

$$\begin{tikzcd} \Gamma_x \arrow[rr, "\Gamma r"] \arrow[dr, "1_{\Gamma_x}"'] & & \Gamma_{\Phi r(x)} \arrow[dl, "\Gamma f"] \\ & \Gamma_x & \end{tikzcd} \qquad \begin{tikzcd} \Gamma_x \arrow[rr, "\Gamma r"] \arrow[dr, "1_{\Gamma_x}"'] & & \Gamma_{\Phi r(x)} \arrow[dl, "\Gamma g"] \\ & \Gamma_x & \end{tikzcd}$$

commute we have that $\varphi\rho = 1_{\coprod_{x\in\Phi(0)}\Gamma_x} = \psi\rho$. it follows from Lemma 1.12 that the generated relation is reflexive.

Similarly induce $\sigma : \coprod_{y\in\Phi(1)}\Gamma_y \to \coprod_{y\in\Phi(1)}\Gamma_y$ such that for every $y \in \Phi(1)$ the diagram

$$\begin{tikzcd} \Gamma_y \arrow[r, "\Gamma s"] \arrow[d, "i_y"'] & \Gamma_{\Phi s(x)} \arrow[d, "i_{\Phi s(y)}"] \\ \coprod_{y\in\Phi(1)}\Gamma_y \arrow[r, "\sigma"'] & \coprod_{y\in\Phi(1)}\Gamma_y \end{tikzcd}$$

commutes. It is easy to show that the diagram

$$\begin{tikzcd} & \coprod_{y\in\Phi(1)}\Gamma_y \arrow[dl, "\varphi"'] \arrow[d, "\sigma"] \arrow[dr, "\psi"] & \\ \coprod_{x\in\Phi(0)}\Gamma_x & \coprod_{y\in\Phi(1)}\Gamma_y \arrow[l, "\psi"] \arrow[r, "\varphi"'] & \coprod_{x\in\Phi(0)}\Gamma_x \end{tikzcd}$$

commutes. Then by Lemma 1.13 the generated relation is symmetric.

For $x \in \Phi(0)$ denote by $\Phi(2)_x$ the set $\{(y_1, y_2) \in \Phi(2) | \Phi f(y_2) = x\}$. By Lemma 1.10 we have that

$$\begin{tikzcd} \coprod_{(y_1,y_2)\in\Phi(2)_x}\Gamma_{y_1y_2} \arrow[r, "\langle i_{y_2}\Gamma p_{12}\rangle"] \arrow[d, "\langle i_{y_1}\Gamma p_{01}\rangle"'] & \coprod_{y\in\Phi f^{-1}(x)}\Gamma_y \arrow[d, "\langle\Gamma f\rangle"] \\ \coprod_{y\in\Phi g^{-1}(x)}\Gamma_y \arrow[r, "\langle\Gamma g\rangle"'] & \Gamma_x \end{tikzcd}$$

is a pullback. It follows by Lemma 1.11 that

$$\begin{tikzcd} \coprod_{(y_1,y_2)\in\Phi(2)}\Gamma_{y_1y_2} \arrow[r, "\langle i_{y_2}\Gamma p_{12}\rangle"] \arrow[d, "\langle i_{y_1}\Gamma p_{01}\rangle"'] & \coprod_{y\in\Phi(1)}\Gamma_y \arrow[d, "\langle\Gamma f\rangle"] \\ \coprod_{y\in\Phi(1)}\Gamma_y \arrow[r, "\langle\Gamma g\rangle"'] & \coprod_{x\in\Phi(0)}\Gamma_x \end{tikzcd}$$

is a pullback. So induce $\tau : \coprod_{(y_1,y_2)\in\Phi(2)} \Gamma_{y_1y_2} \to \coprod_{y\in\Phi(1)} \Gamma_y$ such that the diagram

$$\begin{tikzcd}
\Gamma_{y_1} \times_P \Gamma_{y_2} \arrow[r, "\Gamma t"] \arrow[d, "i_{(y_1,y_2)}"'] & \Gamma_{\Phi t(y_1,y_2)} \arrow[d, "i_{\Phi t(y_1,y_2)}"] \\
\coprod_{(y_1,y_2)\in\Phi(2)} \Gamma_{y_1y_2} \arrow[r, "\tau"'] & \coprod_{y\in\Phi(1)} \Gamma_y
\end{tikzcd}$$

commutes for every $(y_1, y_2) \in \Phi(2)$. It is easy to see that the diagram

$$\begin{tikzcd}
\coprod_{y\in\Phi(1)} \Gamma_y \arrow[d, "\psi"'] & \coprod_{(y_1,y_2)\in\Phi(2)} \Gamma_{y_1y_2} \arrow[l, "\langle i_{y_2}\Gamma p_{12}\rangle"'] \arrow[r, "\langle i_{y_1}\Gamma p_{01}\rangle"] \arrow[d, "\tau"] & \coprod_{y\in\Phi(1)} \Gamma_y \arrow[d, "\varphi"] \\
\coprod_{x\in\Phi(0)} \Gamma_x & \coprod_{y\in\Phi(1)} \Gamma_y \arrow[l, "\psi"] \arrow[r, "\varphi"'] & \coprod_{x\in\Phi(0)} \Gamma_x
\end{tikzcd}$$

commutes. Then by Lemma 1.14 the generated relation is transitive. □

Now, for a left exact category $\boldsymbol{C}$ the objects of $F(\boldsymbol{C})$ are pairs of models

$$(\mathcal{S} \xrightarrow{\Phi} \boldsymbol{Set}_0, \mathcal{S}_\Phi \xrightarrow{\Gamma} \boldsymbol{C}).$$

We are thinking that the pair (Φ, Γ) represents the quotient of the equivalence relation generated by $\coprod_{y\in\Phi(1)} \Gamma_y \underset{\langle i_{\Phi g(y)}\Gamma g\rangle}{\overset{\langle i_{\Phi f(y)}\Gamma f\rangle}{\rightrightarrows}} \coprod_{x\in\Phi(0)} \Gamma_x$, but this is not in $\boldsymbol{C}$ since we are only asking for finite limits in $\boldsymbol{C}$.

Now, for the arrows in $F(\boldsymbol{C})$ we need to retain only the information given by f and g. To do this we consider the graph $\boldsymbol{H} = 1 \underset{g}{\overset{f}{\rightrightarrows}} 0$ and regard it as a limit sketch where the set of commutative diagrams and the set of limit diagrams are both empty. That is, we consider the sketch $\mathcal{T} = (\boldsymbol{H}, \emptyset, \emptyset)$. We have an obvious sketch arrow $i : \mathcal{T} \to \mathcal{S}$. We are also going to use the sketch $\mathcal{I} = (1, \emptyset, \emptyset)$ and the sketch morphisms $\mathcal{I} \underset{1}{\overset{0}{\rightrightarrows}} \mathcal{T}$. Given a model $\Phi : \mathcal{S} \to \boldsymbol{Set}_0$ we can define the graph $\boldsymbol{H}_\Phi$ whose set of nodes is $\Phi(1) \coprod \Phi(0)$ and with arrows $f : y \to \Phi f(y)$ and $g : y \to \Phi g(y)$ for every $y \in \Phi(1)$. Then let $\mathcal{T}_\Phi = (\boldsymbol{H}_\Phi, \emptyset, \emptyset)$. In the same fashion let $\mathcal{I}_{\Phi 0} = (\boldsymbol{H}_{\Phi 0}, \emptyset, \emptyset)$ and $\mathcal{I}_{\Phi 1} = (\boldsymbol{H}_{\Phi 1}, \emptyset, \emptyset)$ where $\boldsymbol{H}_{\Phi 0}$ is the discrete graph with nodes $\Phi(0)$ and $\boldsymbol{H}_{\Phi 1}$ is the discrete graph with nodes $\Phi(1)$. We have the obvious sketch arrows $\mathcal{T}_\Phi \to \mathcal{S}_\Phi, \mathcal{I}_{\Phi 0} \to \mathcal{T}_\Phi$ and $\mathcal{I}_{\Phi 1} \to \mathcal{T}_\Phi$.

Given models $\Phi, \Psi : \mathcal{S} \to \boldsymbol{Set}_0$, an arrow $h : \Phi i \to \Psi i$ of models induces an obvious $\hat{h} : \mathcal{T}_\Phi \to \mathcal{T}_\Psi$. Suppose we have two pairs of models $(\mathcal{S} \xrightarrow{\Phi} \boldsymbol{Set}_0, \mathcal{S}_\Phi \xrightarrow{\Gamma} \boldsymbol{C})$ and $(\mathcal{S} \xrightarrow{\Psi} \boldsymbol{Set}_0, \mathcal{S}_\Psi \xrightarrow{\Delta} \boldsymbol{C})$ and a pair of arrows of models

(1.3)
$$
\begin{tikzcd}
\mathcal{T} \arrow[r, "\mathtt{i}"] \arrow[d, "\mathtt{i}"'] & \mathcal{S} \arrow[d, "\Phi"] \arrow[dl, phantom, "h\swarrow"] \\
\mathcal{S} \arrow[r, "\Psi"'] & \boldsymbol{Set}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T}_\Phi \arrow[r] \arrow[d, "\hat{h}"'] & \mathcal{S}_\Phi \arrow[dr, "\Gamma"] \arrow[d, phantom, "\sigma\Downarrow"] & \\
 & & \boldsymbol{C} \\
\mathcal{T}_\Psi \arrow[r] & \mathcal{S}_\Psi \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

Let's take a closer look at what these arrows are. h is a pair of functions making the diagram

$$
\begin{tikzcd}
\Phi(1) \arrow[r, shift left, "\Phi(f)"] \arrow[r, shift right, "\Phi(g)"'] \arrow[d, "h0"'] & \Phi(0) \arrow[d, "h1"] \\
\Psi(1) \arrow[r, shift left, "\Psi(f)"] \arrow[r, shift right, "\Psi(g)"'] & \Psi(0)
\end{tikzcd}
$$

sequentially commutative. Then σ gives an arrow $\sigma x : \Gamma_x \to \Delta_{h0(x)}$ in $\boldsymbol{C}$ for every $x \in \Phi(0)$ and an arrow $\sigma y : \Gamma_y \to \Delta_{h1(y)}$ in $\boldsymbol{C}$ for every $y \in \Phi(1)$ in such a way that the diagram

$$
\begin{tikzcd}
\Gamma_{\Phi g(y)} \arrow[d, "\sigma\Phi g(y)"'] & \Gamma_y \arrow[l, "\Gamma g"'] \arrow[r, "\Gamma f"] \arrow[d, "\sigma y"] & C_{\Phi f(y)} \arrow[d, "\sigma\Phi f(y)"] \\
\Delta_{\Phi g(h1(y))} & \Delta_{h1(y)} \arrow[l, "\Delta g"] \arrow[r, "\Delta f"'] & \Delta_{\Phi f(h1(y))}
\end{tikzcd}
$$

commutes for all $y \in \Phi(1)$. What this represents in our informal discussion is a sequentially commutative diagram

$$
\begin{tikzcd}
\coprod_{\Phi(1)} \Gamma_y \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\langle i_{h1(y)}\sigma y\rangle"'] & \coprod_{\Phi(0)} \Gamma_x \arrow[d, "\langle i_{h0(x)}\sigma x\rangle"] \\
\coprod_{\Psi(1)} \Delta_{y'} \arrow[r, shift left] \arrow[r, shift right] & \coprod_{\Psi(1)} \Delta_{x'}
\end{tikzcd}
$$

that would induce an arrow between the coequalizers. There is, of course, no unique way to induce arrows between coequalizers so we will need equivalence classes. The definition is as follows.

Given a left exact category $\boldsymbol{C}$ let $F(\boldsymbol{C})$ be the category whose objects are pairs of models $(\mathcal{S} \xrightarrow{\Phi} \boldsymbol{Set}_0,\ \mathcal{S}_\Phi \xrightarrow{\Gamma} \boldsymbol{C})$. A morphism

$$(\mathcal{S} \xrightarrow{\Phi} \boldsymbol{Set}_0, \mathcal{S}_\Phi \xrightarrow{\Gamma} \boldsymbol{C}) \to (\mathcal{S} \xrightarrow{\Psi} \boldsymbol{Set}_0, \mathcal{S}_\Psi \xrightarrow{\Delta} \boldsymbol{C})$$

is an equivalence class $[(h,\sigma)]$ such that (h,σ) are as in 1.3. The equivalence relation is defined as follows, $(h,\sigma) \sim (k,\tau)$ if there exist morphisms of models d and δ

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "0"] \arrow[d, "1"'] & \mathcal{S} \arrow[d, "\Phi"] \arrow[dl, phantom, "\swarrow d"] \\
\mathcal{S} \arrow[r, "\Psi"'] & \boldsymbol{Set}_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_{\Phi 0} \arrow[r] \arrow[d, "d"'] & \mathcal{S}_\Phi \arrow[d, "\delta"'] \arrow[dr, "\Gamma"] & \\
& & \boldsymbol{C} \\
\mathcal{I}_{\Psi 1} \arrow[r] & \mathcal{S}_\Psi \arrow[ur, "\Delta"'] &
\end{tikzcd} \tag{1.4}$$

such that the following diagrams

$$\begin{tikzcd}
\Psi(0) & \Phi(0) \arrow[l, "k0"'] \arrow[r, "h0"] \arrow[d, "d"] & \Psi(0) \\
& \Psi(1) \arrow[ul, "\Psi g"] \arrow[ur, "\Psi f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta_{k0(x)} & \Gamma_x \arrow[l, "\tau x"'] \arrow[r, "\sigma x"] \arrow[d, "\delta x"] & \Delta_{h0(x)} \\
& \Delta_{d(x)} \arrow[ul, "\Delta g"] \arrow[ur, "\Delta f"'] &
\end{tikzcd} \tag{1.5}$$

commute. We show that $\sim$ is an equivalence relation. Given (h,σ) define $d = (\Phi(0) \xrightarrow{h0} \Psi(0) \xrightarrow{\Psi r} \Psi(1))$ and for every $x \in \Phi(0)$ define δx as the composition

$$\Gamma_x \xrightarrow{\sigma x} \Delta_{h0(x)} \xrightarrow{\Delta r} \Delta_{\Phi r(h0(x))}.$$

With these definitions it is clear that $(h,\sigma) \sim (h,\sigma)$. Suppose now that $(h,\sigma) \sim (k,\tau)$, then there exist d and δ with the corresponding properties above. Define $d' = (\Phi(0) \xrightarrow{d} \Psi(1) \xrightarrow{\Psi s} \Psi(1))$, and $\delta'(x \in \Phi(0))$ as the composition

$$\Gamma_x \xrightarrow{\delta x} \Delta_{\ d(x)} \xrightarrow{\Delta s} \Delta_{\Psi s(h0(x))}.$$

It is not hard to see that d' and δ' satisfy the conditions for $(k,\tau) \sim (h,\sigma)$. Suppose now that $(h,\sigma) \sim (k,\tau)$ and $(k,\tau) \sim (l,\theta)$, with d and δ guaranteeing the first equivalence and d', δ' the second. Then there exists a unique arrow $\Phi(0) \to \Psi(2)$ that makes the diagram

$$\begin{tikzcd}
\Phi(0) \arrow[dr] \arrow[drr, bend left, "d'"] \arrow[ddr, bend right, "d"'] & & \\
& \Psi(2) \arrow[r, "\Psi p_{01}"] \arrow[d, "\Psi p_{12}"'] & \Psi(1) \arrow[d, "\Psi f"] \\
& \Psi(1) \arrow[r, "\Psi g"'] & \Psi(0)
\end{tikzcd}$$

commute. For every $x \in \Phi(0)$ there exists a unique arrow $\Gamma_x \to \Delta_{d(x)d'(x)}$ that makes the diagram

$$\begin{tikzcd}
\Gamma_x \arrow[dr] \arrow[drr, bend left, "\delta' x"] \arrow[ddr, bend right, "\delta x"'] & & \\
& \Delta_{d(x)d'(x)} \arrow[r, "\Delta p_{12}"] \arrow[d, "\Delta p_{01}"'] & \Delta_{d'(x)} \arrow[d, "\Delta f"] \\
& \Delta_{d(x)} \arrow[r, "\Delta g"'] & \Delta_{\Psi f d'(x)}
\end{tikzcd}$$

commute. Define $d'' = (\Phi(0) \to \Psi(2) \xrightarrow{\Psi t} \Psi(1))$, and for every $x \in \Phi(0)$, define $\delta'' x$ as the composition

$$\Gamma_x \to \Delta_{d(x)d'(x)} \xrightarrow{\Delta t} \Delta_{\Psi t(d(x),d'(x))}.$$

It is easy then to show that $(h, \sigma) \sim (l, \theta)$.

Composition in $F(\boldsymbol{C})$ is defined as follows. Given

$$(\Phi, \Gamma) \xrightarrow{[(h,\sigma)]} (\Psi, \Delta) \xrightarrow{[(k,\tau)]} (\Upsilon, \Xi)$$

its composition is simply $[(kh, \tau\sigma)]$. It is not hard to prove that the composition is well defined. It is clearly associative and the identity morphism of (Φ, Γ) is $[(1,1)]$.

If $\boldsymbol{P}$ is a pretopos we know from Proposition 1.26 that for any object $(\mathcal{S} \xrightarrow{\Phi} \boldsymbol{Set}_0, \mathcal{S}_\Phi \xrightarrow{\Gamma} \boldsymbol{P})$ in $F\boldsymbol{P}$ we obtain a pair of arrows (see 1.2) $\coprod_{\Phi(1)} \Gamma_y \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \coprod_{\Phi(0)} \Gamma_x$ whose generated relation is an equivalence relation. This in particular means that the pair of arrows has a coequalizer $\coprod_{\Phi(1)} \Gamma_y \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \coprod_{\Phi(0)} \Gamma_x \xrightarrow{u} U$ (the quotient of the generated equivalence relation). Given a pair (h,σ) as in 1.3 we obtain a commutative diagram

$$\begin{tikzcd} \coprod_{\Phi(1)} \Gamma_y \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] \arrow[d, "\langle i_{h1(y)}\sigma y\rangle"'] & \coprod_{\Phi(0)} \Gamma_x \arrow[r, "u"] \arrow[d, "\langle i_{h0(x)}\sigma x\rangle"'] & U \arrow[d, dotted, "t_{(h,\sigma)}"] \\ \coprod_{\Psi(1)} \Delta_{y'} \arrow[r, shift left, "\varphi'"] \arrow[r, shift right, "\psi'"'] & \coprod_{\Psi(0)} \Delta_{x'} \arrow[r, "u'"] & U' \end{tikzcd}$$

therefore we can induce $t_{(h,\sigma)}$ above making the diagram commutative.

Proposition 1.27. *With the above notation, if $(h,\sigma) \sim (k,\tau)$ then $t_{(h,\sigma)} = t_{(k,\tau)}$*

Proof. Let d and δ be as in 1.4 such that the corresponding diagrams commute making $(h,\sigma) \simeq (k,\tau)$. Consider the arrow $\coprod_{\Phi(0)} \Gamma_x \xrightarrow{\langle i_{d(x)}\delta x\rangle} \coprod_{\Psi(1)} \Delta_{y'}$. Using the commutativity of 1.5 we have that the diagram

$$\begin{tikzcd} \coprod_{\Psi(0)} \Delta_{x'} & \coprod_{\Phi(0)} \Gamma_x \arrow[l, "\langle i_{k0(x)}\tau x\rangle"'] \arrow[r, "\langle i_{h0(x)}\sigma x\rangle"] \arrow[d, "\langle i_{d(x)}\delta x\rangle"] & \coprod_{\Psi(0)} \Delta_{x'} \\ & \coprod_{\Psi(1)} \Delta_{y'} \arrow[ul, "\psi'"] \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

commutes. Since u coequalizes (φ',ψ') it follows that

$$\coprod_{\Phi(0)} \Gamma_x \underset{\langle i_{k0(x)}\tau x\rangle}{\overset{\langle i_{h0(x)}\sigma x\rangle}{\rightrightarrows}} \coprod_{\Psi(0)} \Delta_{x'} \xrightarrow{u'} U$$

commutes. Therefore $\coprod_{\Phi(0)} \Gamma_x \xrightarrow{u} U \underset{t_{(k,\tau)}}{\overset{t_{(h,\sigma)}}{\rightrightarrows}} U'$ also commutes. Since u is epi we are done. □

Proposition 1.28. *For any small left exact category $\boldsymbol{C}$ the category $F\boldsymbol{C}$ is equivalent to the category $(\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh}$.*

Proof. Define $G : F\boldsymbol{C} \to (\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh}$ such that any object (Φ,Γ) in $F\boldsymbol{C}$ the diagram

$$\coprod_{\Phi(1)} \boldsymbol{C}(\_,\Gamma_y) \underset{\langle i_{\Phi g(y)}\boldsymbol{C}(\_,\Gamma g)\rangle}{\overset{\langle i_{\Phi f(y)}\boldsymbol{C}(\_,\Gamma f)\rangle}{\rightrightarrows}} \coprod_{\Phi(0)} \boldsymbol{C}(\_,\Gamma_x)) \longrightarrow G(\Phi,\Gamma)$$

is a coequalizer. The coequalizer exists as a consequence of Proposition 1.26. Given $[(h,\sigma)] : (\Phi,\Gamma) \to (\Psi,\Delta)$ define $G([(h,\sigma)])$ as the induced arrow such that

$$\begin{tikzcd}
\coprod_{\Phi(0)} \boldsymbol{C}(-,\Gamma_x) \arrow[r] \arrow[d, "\langle i_{h0(x)}(\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh}(-,\boldsymbol{C}(-,\sigma x))\rangle"'] & G(\Phi,\Gamma) \arrow[d, "G([(h,\sigma)])"] \\
\coprod_{\Psi(0)} \boldsymbol{C}(-,\Delta_{x'}) \arrow[r] & G(\Psi,\Delta)
\end{tikzcd}$$

commutes. It follows from Proposition 1.5 that $G[(h,\sigma)]$ is well defined.

In the other direction define $H : (\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh} \to \boldsymbol{FC}$ as follows. For every K in $(\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh}$ choose a finite set $\Phi(0)$, an object Γ_x for every $x \in \Phi(0)$ and an epimorphism $\coprod_{\Phi(0)} \boldsymbol{C}(-,\Gamma_x) \twoheadrightarrow K$. Consider $R \underset{r_2}{\overset{r_1}{\rightrightarrows}} \coprod_{\Phi(0)} \boldsymbol{C}(-,\Gamma_x)$, kernel pair of this epimorphism. Since R is compact we can choose a finite set $\Phi(1)$, an object Γ_y in $\boldsymbol{C}$ for every $y \in \Phi(1)$ and an epimorphism $\coprod_{\Phi(1)} \boldsymbol{C}(-,\Gamma_y) \twoheadrightarrow R$. We obtain then a pair of arrows

$$\coprod_{\Phi(1)} \boldsymbol{C}(-,\Gamma_y) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \coprod_{\Phi(0)} \boldsymbol{C}(-,\Gamma_x)$$

whose generated relation is the equivalence relation (r_1, r_2). We can then find functions $\Phi f, \Phi g : \Phi(1) \to \Phi(0)$ and arrows $\Gamma_{\Phi f(y)} \xleftarrow{\Gamma f} \Gamma_y \xrightarrow{\Gamma g} \Gamma_{\Phi g(y)}$ for every $y \in \Phi(1)$ such that the diagrams

$$\begin{tikzcd}
\boldsymbol{C}(-,\Gamma_y) \arrow[r, "i_y"] \arrow[d, "\boldsymbol{C}(-,\Gamma f)"'] & \coprod_{\Phi(1)} \boldsymbol{C}(-,\Gamma_y) \arrow[d, "\phi"] \\
\boldsymbol{C}(-,\Gamma_{\Phi f(y)}) \arrow[r, "i_{\Phi f(y)}"'] & \coprod_{\Phi(0)} \boldsymbol{C}(-,\Gamma_x)
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{C}(-,\Gamma_y) \arrow[r, "i_y"] \arrow[d, "\boldsymbol{C}(-,\Gamma g)"'] & \coprod_{\Phi(1)} \boldsymbol{C}(-,\Gamma_y) \arrow[d, "\psi"] \\
\boldsymbol{C}(-,\Gamma_{\Phi g(y)}) \arrow[r, "i_{\Phi g(y)}"'] & \coprod_{\Phi(0)} \boldsymbol{C}(-,\Gamma_x)
\end{tikzcd}$$

commute. Since (r_1, r_2) is reflexive and $\coprod_{\Phi(1)} \boldsymbol{C}(-,\Gamma_y) \twoheadrightarrow R$ epimorphic we can choose a function $\Phi r : \Phi(0) \to \Phi(1)$ and arrows $\Gamma r : C_x \to D_{\Phi r(x)}$ such that the diagrams

$$\begin{tikzcd}
\Phi(0) & \Phi(0) \arrow[l, "1"'] \arrow[r, "1"] \arrow[d, "\Phi r"] & \Phi(0) \\
 & \Phi(1) \arrow[ul, "\Phi f"] \arrow[ur, "\Phi g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma_x & \Gamma_x \arrow[l, "1_{\Gamma_x}"'] \arrow[r, "1_{\Gamma_x}"] \arrow[d, "\Gamma r"] & \Gamma_x \\
 & \Gamma_{\Phi r(x)} \arrow[ul, "\Gamma f"] \arrow[ur, "\Gamma g"'] &
\end{tikzcd}$$

commute

Similarly, using symmetry and transitivity we can define the rest of the elements necessary to obtain an object (Φ, Γ) of FC. Define then $H(K) = (\Phi, \Gamma)$. Given an arrow $\mu : K \to K'$ in $(\boldsymbol{Set}^{C^{op}})_{coh}$, assume $H(K') = (\Psi, \Delta)$. Since $\coprod_{\Psi(0)} C(\_, \Delta_{x'}) \twoheadrightarrow K'$ is epimorphic there exists a map $K \to \coprod_{\Psi(0)} C(\_, \Delta_{x'})$ such that

$$\begin{tikzcd} K \arrow[rr, "\mu"] \arrow[dr] & & K' \\ & \coprod_{\Psi(0)} C(\_, \Delta_{x'}) \arrow[ur] & \end{tikzcd}$$

commutes. This induces an arrow

$$\coprod_{\Phi(0)} C(\_, \Gamma_x) \longrightarrow \coprod_{\Psi(0)} C(\_, \Delta_{x'}).$$

Therefore we can find a function $h0 : \Phi(0) \to \Psi(0)$ and arrows $\sigma x : \Gamma_x \to \Delta_{h0(x)}$ for every $x \in \Phi(0)$ such that the diagram

$$\begin{tikzcd} \coprod_{\Phi(0)} C(\_, \Gamma_x) \arrow[r, "\langle i_{h0(x)}\sigma x\rangle"] \arrow[d] & \coprod_{\Psi(0)} C(\_, \Delta_{x'}) \arrow[d] \\ K \arrow[r, "\mu"'] & K' \end{tikzcd}$$

commutes. There exists then an arrow $R \to R'$ such that the diagram

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{\Phi(0)} C(\_, \Gamma_x) \arrow[d] \\ R' \arrow[r, shift left] \arrow[r, shift right] & \coprod_{\Psi(0)} C(\_, \Delta_{x'}) \end{tikzcd}$$

is sequentially commutative. Since $\coprod_{\Psi(1)} C(\_, \Delta_{y'}) \twoheadrightarrow R$ is an epimorphism we can find an arrow $\coprod_{\Phi(1)} C(\_, \Gamma_y) \to \coprod_{\Psi(1)} C(\_, \Delta_{y'})$ such that the diagram

$$\begin{tikzcd} \coprod_{\Phi(1)} C(\_, \Gamma_y) \arrow[r] \arrow[d] & \coprod_{\Psi(1)} C(\_, \Delta_{y'}) \arrow[d] \\ R \arrow[r] & R' \end{tikzcd}$$

commutes. This gives a function $h1 : \Phi(1) \to \Psi(1)$ and arrows $\sigma y : \Gamma_y \to \Delta_{h1(y)}$ for every $y \in \Phi(1)$ such that

$$\begin{tikzcd}
\boldsymbol{C}(_, \Gamma_y) \arrow[r, "\boldsymbol{C}(_, \sigma y)"] \arrow[d, "i_y"'] & \boldsymbol{C}(_, \Delta_{h1(y)}) \arrow[d, "i_{h1(y)}"] \\
\coprod_{\Phi(1)} \boldsymbol{C}(_, \Gamma_y) \arrow[r] & \coprod_{\Psi(1)} \boldsymbol{C}(_, \Delta_{y'})
\end{tikzcd}$$

commutes. It is easy to show that h and σ as defined above are arrows of sketches as in 1.3. Define $H(\mu) = [(h, \sigma)]$. It is not hard to see that if we change the choices made above to produce (h, σ) we obtain an equivalent pair. G is the pseudo-inverse of H $\square$

Chapter 2

Ultracategories

The concepts of pre-ultracategory, ultramorphism, ultracategory and Makkai's theorem (Theorem 2.3) all are taken from [15].

Given a pretopos $\boldsymbol{P}$ we want to consider the category $\boldsymbol{Mod}(\boldsymbol{P})$ of models of $\boldsymbol{P}$. $\boldsymbol{Mod}(\boldsymbol{P})$ has filtered colimits (and they are calculated pointwise) but in general we can not guarantee the existence of any other kind of colimits. The situation for limits in $\boldsymbol{Mod}(\boldsymbol{P})$ is even worse. However, $\boldsymbol{Mod}(\boldsymbol{P})$ has ultraproducts and they are pointwise. That is, given an ultrafilter $(I,\mathcal{U})$ (a set I with an ultrafilter $\mathcal{U}$ on I) we have that for every family $\langle M_i \rangle_I$ of models of $\boldsymbol{P}$ the ultraproduct $\varinjlim_{J\in\mathcal{U}} \prod_{j\in J} M_j$ is a model of $\boldsymbol{P}$, where the products and the filtered colimit are taken in $\boldsymbol{Set}^{\boldsymbol{P}}$. So we have a functor $[\mathcal{U}] : (\boldsymbol{Mod}(\boldsymbol{P}))^I \to \boldsymbol{Mod}(\boldsymbol{P})$ that assigns to any I-family of models its ultraproduct. Pre-ultracategories are an attempt to capture this situation.

2.1 Pre-Ultracategories

Definition 2.1. A pre-ultracategory $\underline{\boldsymbol{A}}$ consists of a category $\boldsymbol{A}$ together with a functor $[\mathcal{U}]_{\underline{\boldsymbol{A}}} : \boldsymbol{A}^I \to \boldsymbol{A}$ for every ultrafilter $(I,\mathcal{U})$. We refer to the functor $[\mathcal{U}]_{\underline{\boldsymbol{A}}}$ as the ultraproduct functor associated to $\mathcal{U}$ in $\underline{\boldsymbol{A}}$.

Given pre-ultracategories $\underline{\boldsymbol{A}}$ and $\underline{\boldsymbol{B}}$, a pre-ultrafunctor $\underline{F} : \underline{\boldsymbol{A}} \to \underline{\boldsymbol{B}}$ is a functor

$F : \boldsymbol{A} \to \boldsymbol{B}$ together with a natural isomorphism $[\mathcal{U}, \underline{F}]$

$$\begin{tikzcd} \boldsymbol{A}^I \arrow[r, "{[\mathcal{U}]_{\underline{\boldsymbol{A}}}}"] \arrow[d, "F^I"'] & \boldsymbol{A} \arrow[d, "F"] \arrow[dl, phantom, "{[\mathcal{U}, \underline{F}]} \swarrow"] \\ \boldsymbol{B}^I \arrow[r, "{[\mathcal{U}]_{\underline{\boldsymbol{B}}}}"'] & \boldsymbol{B} \end{tikzcd}$$

for every ultrafilter $(I, \mathcal{U})$. Pre-ultrafunctors compose in the obvious way.

Given pre-ultrafunctors $\underline{F}, \underline{G} : \underline{\boldsymbol{A}} \to \underline{\boldsymbol{B}}$, a pre-ultranatural transformation $\underline{\tau} : \underline{F} \to \underline{G}$ is a natural transformation $\tau : F \to G : \boldsymbol{A} \to \boldsymbol{B}$ such that

$$\begin{tikzcd} F \circ [\mathcal{U}]_{\underline{\boldsymbol{A}}} \arrow[r, "{\tau[\mathcal{U}]_{\underline{\boldsymbol{A}}}}"] \arrow[d, "{[\mathcal{U}, F]}"'] & G \circ [\mathcal{U}]_{\underline{\boldsymbol{A}}} \arrow[d, "{[\mathcal{U}, G]}"] \\ {[\mathcal{U}]_{\underline{\boldsymbol{B}}} \circ F^I} \arrow[r, "{[\mathcal{U}]_{\underline{\boldsymbol{B}}}\tau^I}"'] & {[\mathcal{U}]_{\underline{\boldsymbol{B}}} \circ G^I} \end{tikzcd}$$

commutes. Pre-ultranatural transformations also compose in the obvious way.

Let $\boldsymbol{PUC}$ denote the 2-category of pre-ultracategories, pre-ultrafunctors and pre-ultranatural transformations whose underlying categories are categories in the second universe.

Whenever we have a pre-ultracategory $\underline{\boldsymbol{A}}$, an ultrafilter $(I, \mathcal{U})$ and a family $\langle A_i \rangle_I$ in $\boldsymbol{A}^I$ we denote $[\mathcal{U}]_{\underline{\boldsymbol{A}}} \langle A_i \rangle$ by $\prod_I A_i/\mathcal{U}$ or sometimes by $\prod A_i/\mathcal{U}$. Similarly, if $\langle f_i \rangle$ is a morphism in $\boldsymbol{A}^I$ we have $[\mathcal{U}]_{\underline{\boldsymbol{A}}} \langle f_i \rangle = \prod_I f_i/\mathcal{U}$.

If $\boldsymbol{P}$ is a pretopos then $\boldsymbol{Mod}(\boldsymbol{P})$ is clearly a pre-ultracategory $\underline{\boldsymbol{Mod}}(\boldsymbol{P})$ with the usual ultraproduct functors. In particular we can consider the pre-ultracategory $\underline{\boldsymbol{Set}}$ of sets together with the usual ultraproduct functors.

2.2 Ultragraphs and Ultramorphisms

The ultraproduct defined above for models is a combination of limits and colimits, therefore we are in very short supply of canonical maps in or out of an ultraproduct (as oppose to an honest limit or colimit). Here is where ultramorphisms try to fix this lack. But before considering the concept of ultramorphism we need the concept of ultragraphs. Ultragraphs are to ultraproducts what limit sketches are to limits. That is, in an ultragraph we want to specify nodes that will represent the ultraproduct of other nodes (the same way as we want some nodes in a limit sketch to represent the limit of some other nodes).

Definition 2.2. An ultragraph $\underline{\boldsymbol{G}}$ is a graph $\boldsymbol{G}$ together with a partition $\boldsymbol{G}^f \cup \boldsymbol{G}^b$ of the nodes of $\boldsymbol{G}$ and such that for every $\beta \in \boldsymbol{G}^b$ we have assigned a triple $(I_\beta, \mathcal{U}_\beta, g_\beta)$ where $(I_\beta, \mathcal{U}_\beta)$ is an ultrafilter and $g_\beta : I_\beta \to \boldsymbol{G}^f$ is a function. The nodes in $\boldsymbol{G}^f$ are called free nodes and the nodes in $\boldsymbol{G}^b$ are called bound nodes.

Then an ultradiagram is the equivalent of a model of a limit sketch. That is, an ultradiagram is a diagram that assigns to a bound node an ultraproduct of the images of the nodes associated with the bound node.

Definition 2.3. Given a pre-ultracategory $\underline{\boldsymbol{A}}$ and an ultragraph $\underline{\boldsymbol{G}}$, an ultradiagram $\underline{D} : \underline{\boldsymbol{G}} \to \underline{\boldsymbol{A}}$ is a diagram $D : \boldsymbol{G} \to \boldsymbol{A}$ together with an isomorphism

$$D(\beta) \xrightarrow{d_\beta} \textstyle\prod_{I_\beta} D(g_\beta(i))/\mathcal{U}_\beta$$

for every $\beta \in \boldsymbol{G}^b$.

Given ultradiagrams $\underline{D}, \underline{D}' : \underline{\boldsymbol{G}} \to \underline{\boldsymbol{A}}$ a morphism $\underline{\sigma} : \underline{D} \to \underline{D}'$ is a natural transformation $\sigma : D \to D'$ between diagrams such that the square

$$\begin{tikzcd}
D(\beta) \arrow[r, "d_\beta"] \arrow[d, "\sigma\beta"'] & \prod_{I_\beta} D(g_\beta(i))/\mathcal{U}_\beta \arrow[d, "\prod_{I_\beta} \sigma(g_\beta(i))/\mathcal{U}_\beta"] \\
D'(\beta) \arrow[r, "d'_\beta"'] & \prod_{I_\beta} D'(g_\beta(i))/\mathcal{U}_\beta
\end{tikzcd}$$

commutes for every $\beta \in \boldsymbol{G}^b$. Morphisms between ultradiagrams compose in the obvious way, so we have a category $\boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{A}})$.

If we have a pre-ultrafunctor $\underline{F} : \underline{\boldsymbol{A}} \to \underline{\boldsymbol{B}}$ and an ultragraph $\underline{\boldsymbol{G}}$ then it is not hard to see that $\underline{F}$ induces a functor $\boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{F}) : \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{A}}) \to \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{B}})$ by composition.

Given a node k in $\underline{\boldsymbol{G}}$ we define the functor $ev_k : \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{A}}) \to \boldsymbol{A}$ as evaluation at k, that is $ev_k(\underline{D}) = D(k)$ and $ev_k(\underline{\sigma}) = \sigma k$ for every $\underline{\sigma} : \underline{D} \to \underline{D}'$ in $\boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{A}})$.

We have the following corollary of Los' theorem 1.4

Corollary 2.1. *For any ultragraph $\underline{\boldsymbol{G}}$ the category $\boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}})$ is a pretopos and the forgetful functor $\boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}}) \to \boldsymbol{Set}^{\boldsymbol{G}}$ is elementary.* $\square$

We are ready now for the definition of ultramorphism.

Definition 2.4. Given a pre-ultracategory $\underline{\boldsymbol{A}}$, an ultragraph $\underline{\boldsymbol{G}}$ and nodes k and l in $\boldsymbol{G}$ an ultramorphism δ of type $(\underline{\boldsymbol{G}}, k, l)$ on $\underline{\boldsymbol{A}}$ is a natural transformation $\delta : ev_k \to ev_l : \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{A}}) \to \boldsymbol{A}$.

An example of an ultramorphism on $\boldsymbol{Set}$ is the following. Let $(I, \mathcal{U})$ be an ultrafilter and $f : I \to J$ be a function. Consider the ultrafilter $\mathcal{V} = \{J_0 \subset J \mid f^{-1}J_0 \in \mathcal{U}\}$ on J. Define the ultragraph $\underline{\boldsymbol{G}}$ as follows. $\underline{\boldsymbol{G}}^b = \{\beta, \gamma\}$ and $\underline{\boldsymbol{G}}^f = J$. There are no arrows in $\underline{\boldsymbol{G}}$. Define $(I_\beta, \mathcal{U}_\beta, g_\beta) = (I, \mathcal{U}, f : I \to J)$ and $(I_\gamma, \mathcal{U}_\gamma, g_\gamma) = (J, \mathcal{V}, id_J)$. We want to induce a natural transformation $\delta : ev_\gamma \to ev_\beta$. Given a family $\langle A_j \rangle_J$ of sets let $\delta\langle A_j \rangle_J : \prod A_j/\mathcal{V} \to \prod A_f(i)/\mathcal{U}$ be the unique map that makes the diagram

$$\begin{tikzcd}
 & \prod_{j\in J_0} A_j \arrow[r, "i_{J_0}"] \arrow[dl, "\pi_{f(i)}"'] \arrow[dd] & \prod A_j/\mathcal{V} \arrow[dd, "\delta\langle A_j\rangle_J"] \\
A_{f(i)} & & \\
 & \prod_{i\in f^{-1}J_0} A_{f(i)} \arrow[r, "i_{f^{-1}J_0}"'] \arrow[ul, "\pi_i"] & \prod A_f(i)/\mathcal{U}
\end{tikzcd}$$

commute for every $J_0 \in \mathcal{V}$. It is not hard to show that δ defined this way is a natural transformation $\delta : ev_\gamma \to ev_\beta$. That is, δ is an ultramorphism. As a particular case observe that when $J = 1$ we obtain the diagonal function $A \to A^{\mathcal{U}}$ for every set A.

Denote by $\Delta\underline{\boldsymbol{Set}}$ the set of all the ultramorphisms on $\underline{\boldsymbol{Set}}$. This makes $\Delta\underline{\boldsymbol{Set}}$ a set in our second universe.

2.3 Ultracategories

Definition 2.5. An ultracategory $\underline{\underline{A}}$ consists of a pre-ultracategory $\underline{A}$ together with an ultramorphism $\delta_{\underline{\underline{A}}} : ev_k \to ev_l : UD(\underline{G}, \underline{A}) \to A$ for every $\delta : ev_k \to ev_l : UD(\underline{G}, \underline{Set}) \to \underline{Set}$ in ΔSet.

Given ultracategories $\underline{\underline{A}}$ and $\underline{\underline{B}}$ an ultrafunctor $\underline{\underline{F}} : \underline{\underline{A}} \to \underline{\underline{B}}$ is a pre-ultrafunctor $\underline{F} : \underline{A} \to \underline{B}$ such that $F\delta_{\underline{\underline{A}}} = \delta_{\underline{\underline{B}}} UD(\underline{G}, \underline{F})$.

Given ultrafunctors $\underline{\underline{F}}, \underline{\underline{G}} : \underline{\underline{A}} \to \underline{\underline{B}}$ an ultranatural transformation $\underline{\underline{\sigma}} : \underline{\underline{F}} \to \underline{\underline{G}}$ is simply a pre-ultranatural transformation $\underline{\sigma} : \underline{F} \to \underline{G}$.

Ultrafunctors and ultranatural transformations compose in the obvious way and we have a 2-category UC whose objects are ultracategories whose underlying pre-ultracategories belong to PUC, ultrafunctors as 1-cells and ultranatural transformations as 2-cells. We have a locally full forgetful functor $UC \to PUC$. When there is no risk of confusion we will omit the corresponding underlining for pre-ultracategories and ultracategories, the context should make clear which one we mean.

If P is a pretopos we can give the pre-ultracategory $\underline{Mod}(P)$ an ultracategory structure as follows. First notice that for every ultragraph $\underline{G}$ and every $P \in P$ we can define the functor $UD(\underline{G}, \underline{Mod}(P)) \to UD(\underline{G}, \underline{Set})$ such that $D \mapsto D(\_)(P)$ and $\sigma \mapsto \sigma(\_)(P)$ for any $\sigma : D \to D'$ in $UD(\underline{G}, \underline{Mod}(P))$ where of course we have that $D(\_)(P)(k) = D(k)(P)$ for any node $k \in \underline{G}$. Given an ultramorphism $\delta : ev_k \to ev_l : UD(\underline{G}, \underline{Set}) \to Set$ define $\delta_{\underline{\underline{Mod}}(P)} : ev_k \to ev_l : UD(\underline{G}, \underline{Mod}(P)) \to Mod(P)$ such that for every $P \in P$ $(\delta_{Mod(P)} D)P = \delta D(\_)P$. In this way we obtain the ultracategory $\underline{\underline{Mod}}(P)$ of models of P.

Proposition 2.2. *For every ultracategory $\underline{\underline{A}}$ the category $UC(\underline{\underline{A}}, \underline{\underline{Set}})$ is a pretopos. Furthermore, the corresponding finite limits and colimits are calculated pointwise.* □

We finally arrive at the main theorem of [15], Makkai's theorem. Let P be a small pretopos. For every $P \in P$ we have that the functor $ev_P : Mod(P) \to Set$ is an ultrafunctor $ev_P : \underline{\underline{Mod}}(P) \to \underline{\underline{Set}}$. This fact allows us to define the functor $ev : P \to UC(\underline{\underline{Mod}}(P), \underline{\underline{Set}})$ such that $P \mapsto ev_P$.

Theorem 2.3. *Given a small pretopos $\boldsymbol{P}$ the functor $ev : \boldsymbol{P} \to \boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ is an equivalence.*

Notice first that according to Lemma 1.15 it suffices to show that $ev : \boldsymbol{P} \to \boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ is subobject full, conservative and that every object in the category $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ has a finite cover via ev. We start with subobject full.

Assume first that we have an object P of $\boldsymbol{P}$ and a monomorphism $\tau : F \to ev_P$ in $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ in which for every model M in $\boldsymbol{Mod}(\boldsymbol{P})$, $\tau M : FM \to MP$ is actual inclusion. Notice that in this case for every ultrafilter $(I, \mathcal{U})$ and any family $\langle M_i \rangle_I$ in $\boldsymbol{Mod}(\boldsymbol{P})^I$ the commutativity of the diagram

$$\begin{tikzcd} F(\prod M_i/\mathcal{U}) \arrow[rr, "{[\mathcal{U}, F]\langle M_i \rangle}"] \arrow[dr, "\tau(\prod M_i/\mathcal{U})"'] & & \prod FM_i/\mathcal{U} \arrow[dl, "\prod \tau M_i/\mathcal{U}"] \\ & \prod M_i P/\mathcal{U} & \end{tikzcd}$$

implies that $[\mathcal{U}, F]\langle M_i \rangle : F(\prod M_i/\mathcal{U}) \to \prod FM_i/\mathcal{U}$ is an identity. Let $\mathcal{S} = \{Q \rightarrowtail P$ in $\boldsymbol{P} | FN \subset NQ$ for every N in $\boldsymbol{Mod}(\boldsymbol{P})\}$

Lemma 2.4. *For every M in $\boldsymbol{Mod}(\boldsymbol{P})$, $FM = \bigcap_{(Q \rightarrowtail P) \in \mathcal{S}} MQ$*

Proof. Let $M \in \boldsymbol{Mod}(\boldsymbol{P})$. Clearly $FM \subset \bigcap_{(Q \rightarrowtail P) \in \mathcal{S}} MQ$. So suppose $a \in \bigcap_{(Q \rightarrowtail P) \in \mathcal{S}} MQ$. Define $\mathcal{T} = \{(Q \rightarrowtail P)$ in $\boldsymbol{P} | a \notin MQ\}$. Clearly $\mathcal{S} \cap \mathcal{T} = \emptyset$, thus for every $(Q \rightarrowtail P) \in \mathcal{T}$ we can choose a model N_Q in $\boldsymbol{Mod}(\boldsymbol{P})$ and an element $b_Q \in FN_Q - N_Q Q$. Observe that $(0 \rightarrowtail P) \in \mathcal{T}$ and if $Q_1 \rightarrowtail P, Q_2 \rightarrowtail P \in \mathcal{T}$ then $Q_1 \vee Q_2 \rightarrowtail P \in \mathcal{T}$. Given $Q \rightarrowtail P \in \mathcal{T}$ define $\uparrow(Q \rightarrowtail P) = \{Q' \rightarrowtail P \in \mathcal{T} | Q \rightarrowtail P \leq Q' \rightarrowtail P$ as subobjects of $P\}$. For any family $\{Q_i \rightarrowtail P\}_{i=1}^n$ of elements of $\mathcal{T}$ we have $\bigcap_{i=1}^n \uparrow(Q_i \rightarrowtail P) = \uparrow(\bigvee_{i=1}^n Q_i \rightarrowtail P)$. Therefore there exists an ultrafilter $\mathcal{U}$ on $\mathcal{T}$ such that for every $Q \rightarrowtail P \in \mathcal{T}$ we have that $\uparrow(Q \rightarrowtail P) \in \mathcal{U}$.

Consider $\langle b_Q \rangle_{\mathcal{T}} \in \prod_{\mathcal{T}} N_Q P/\mathcal{U}$.

Let $R \rightarrowtail P$ in $\boldsymbol{P}$ and assume that $\langle b_Q \rangle \in \prod_{\mathcal{T}} N_Q R/\mathcal{U}$. We want to show that $a \in MR$. Suppose not, then $R \rightarrowtail P \in \mathcal{T}$ and $\uparrow(R \rightarrowtail P) \in \mathcal{U}$. Since $\langle b_Q \rangle_{\mathcal{T}} \in \prod_{\mathcal{T}} N_Q P/\mathcal{U}$ there exists $J \in \mathcal{U}$ such that for every $Q \rightarrowtail P \in J$, $b_Q \in N_Q R$. Since $J \cap \uparrow(R \rightarrowtail P) \in \mathcal{U}$ we have that there exists $(R' \rightarrowtail P) \geq (R \rightarrowtail P)$ such that $b_{R'} \in N_{R'} R$. Since

$N_{R'}R \subset N_{R'}R'$ we have $b_{R'} \in N_{R'}R'$. This is a contradiction, so we can conclude that $a \in MR$.

We have showed that for every $R \rightarrowtail P$, $\langle b_Q \rangle_T \in \prod_T N_Q R/\mathcal{U}$ implies $a \in MR$. Therefore by Theorem 1.21 there exist an ultrafilter $(I, \mathcal{V})$ and an arrow

$$h : \prod_T N_Q/\mathcal{U} \to M^{\mathcal{V}}$$

in $\boldsymbol{Mod}(\boldsymbol{P})$ such that $hP\langle b_Q \rangle = \delta P(a)$ where $\delta : M \to M^{\mathcal{V}}$ is the diagonal. Since $\langle b_Q \rangle \in F(\prod_T N_Q/\mathcal{U})$ we have that $\langle a \rangle_I = \delta P(a) = hP\langle b_Q \rangle \in F(M^{\mathcal{V}}) = (FM)^{\mathcal{V}}$. Therefore there exists $I_0 \in \mathcal{V}$ such that for every $i \in I_0$, $a \in MP$. That is, $a \in MP$.

□

Lemma 2.5. *With the same notation as the previous lemma, there exists $R \rightarrowtail P \in \mathcal{S}$ such that $F = ev_R$.*

Proof. Suppose not. That is, assume that for every $Q \rightarrowtail P \in \mathcal{S}$ there exist a model M_Q in $\boldsymbol{Mod}(\boldsymbol{P})$ and an element $a_Q \in M_Q Q - F(M_Q)$. Now, $(1_P : P \rightarrowtail P) \in \mathcal{S}$ and if $Q_1 \rightarrowtail P, Q_2 \rightarrowtail P \in \mathcal{S}$ then $Q_1 \wedge Q_2 \rightarrowtail P \in \mathcal{S}$. For every $Q \rightarrowtail P \in \mathcal{S}$ define $\downarrow(Q \rightarrowtail P) = \{Q' \rightarrowtail P \in \mathcal{S} | (Q' \rightarrowtail P) \leq (Q \rightarrowtail P)$ as subobjects of $P\}$. We have that $\bigcap_{i=1}^n (\downarrow(Q_i \rightarrowtail P)) = \downarrow(\bigwedge_{i=1}^n Q_i \rightarrowtail P)$. There exists then an ultrafilter $\mathcal{W}$ on $\mathcal{S}$ such that for every $Q \rightarrowtail P \in \mathcal{S}$ we have $\downarrow(Q \rightarrowtail P) \in \mathcal{W}$.

Consider $\langle a_Q \rangle_S \in \prod_S M_Q P/\mathcal{W}$.

Let $R \rightarrowtail P \in \mathcal{S}$. We have that for every $R' \rightarrowtail P \in \downarrow(R \rightarrowtail P)$, $a_{R'} \in M_{R'}R' \subset M_{R'}R$. That is $\langle a_Q \rangle \in \prod_S M_Q R/\mathcal{W}$. Therefore $\langle a_Q \rangle \in \bigcap_{(R \rightarrowtail P) \in \mathcal{S}} \prod_S M_Q R/\mathcal{W}$. So according to the previous lemma we have that $\langle a_Q \rangle \in F(\prod_S M_Q/\mathcal{W}) = \prod_S FM_Q/\mathcal{W}$. This means that we can find $(Q \rightarrowtail P) \in \mathcal{S}$ such that $a_Q \in FM_Q$. This is a contradiction. □

Consider now an arbitrary arrow $\sigma : G \to ev_P$ in $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$. Consider its image

$$\begin{tikzcd} G \arrow[rr, "\sigma"] \arrow[dr, "e"'] & & ev_P \\ & H \arrow[ur, "m"'] & \end{tikzcd}$$

Since images in $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ are pointwise we may assume that for every M in $\boldsymbol{Mod}(\boldsymbol{P})$, $mM : HM \to MP$ is really an inclusion . Then there exists $R \rightarrowtail P$

such that $H = ev_R$. If $\sigma : G \to ev_P$ is a monomorphism we obtain that $e : G \to H$ as above is an isomorphism in $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$. We have proved

Proposition 2.6. *If $\boldsymbol{P}$ is a small pretopos then the functor*

$$ev : \boldsymbol{P} \to \boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$$

is subobject full. □

We turn our attention now to ev being conservative. Given a small pretopos $\boldsymbol{P}$ we can consider the precanonical category J on $\boldsymbol{P}$ and form the category $Sh(\boldsymbol{P}, J)$. Using Theorem 1.4 and Proposition 1.5 we can find I in $\boldsymbol{Set}$ and a surjection

$$\boldsymbol{Set}/I \underset{f_*}{\overset{f^*}{\rightleftarrows}} Sh(\boldsymbol{P}, J).$$

Notice that we need $\boldsymbol{P}$ to be small to apply 1.4. We have then that the composition $\boldsymbol{P} \xrightarrow{y} Sh(\boldsymbol{P}, J) \xrightarrow{f^*} \boldsymbol{Set}/I$ is elementary and conservative, where y is the usual functor.

Proposition 2.7. *If $\boldsymbol{P}$ is a small pretopos then $ev : \boldsymbol{P} \to \boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ is conservative.*

Proof. Suppose we have two subobjects $Q \rightarrowtail P$ and $R \rightarrowtail P$ of an object P in $\boldsymbol{P}$ such that $ev_Q = ev_R$ in $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$. Take the functor $\boldsymbol{P} \xrightarrow{y} Sh(\boldsymbol{P}, J) \xrightarrow{f^*} \boldsymbol{Set}/I$ defined above and define $M_i = (\boldsymbol{P} \xrightarrow{y} Sh(\boldsymbol{P}, J) \xrightarrow{f^*} \boldsymbol{Set}/I \xrightarrow{i^*} \boldsymbol{Set})$ for every $i \in I$. Then for every i in I we have that M_i is in $\boldsymbol{Mod}(\boldsymbol{P})$ and $ev_Q M_i = ev_R M_i$. Therefore $i^* f^* y Q = i^* f^* y R$ for every $i \in I$. Then clearly $f^* y Q = f^* y R$. since $f^* y$ is conservative we conclude that $(Q \rightarrowtail P) = (R \rightarrowtail P)$ as subobjects of P. □

Now we turn our attention to the other part of the proof namely, that every object F in $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ has a finite cover via ev. Let M be a model in $\boldsymbol{Mod}(\boldsymbol{P})$ and $x \in FM$. If we are hoping to find a finite cover for F via ev we should be able to find an ultranatural transformation $\Phi : ev_P \to F$ for some P in $\boldsymbol{P}$ such that $x \in Im(\Phi M)$. That is to say, there exists $a \in MP$ such that $\Phi M(a) = x$. Notice that if this happens then for any two arrows $h, k : M \to N$ in $\boldsymbol{Mod}(\boldsymbol{P})$ we have that if $hP(a) = kP(a)$ then $Fh(x) = Fk(x)$.

Definition 2.6. Given $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$, M in $\boldsymbol{Mod}(\boldsymbol{P})$ and P in $\boldsymbol{P}$ we say that an element $a \in MP$ is a support for an element $x \in FM$ if for every pair of arrows $h, k : M \to N$ in $\boldsymbol{Mod}(\boldsymbol{P})$ we have that $hP(a) = kP(a)$ implies that $Fh(x) = Fk(x)$. We say that $x \in FM$ has a support if there exist an object P in $\boldsymbol{P}$ and an element $a \in MP$ that is a support for $x \in FM$.

We will show that if $a \in MP$ is a support for $x \in FM$ where F is an ultrafunctor then there exist a subobject $Q \rightarrowtail P$ in $\boldsymbol{P}$ with $a \in MQ$ and an ultranatural transformation $\Phi : ev_Q \to F$ such that $\Phi M(a) = x$. Since we already know that every subobject of ev_P in $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$ is of the form ev_Q for some subobject Q of P in $\boldsymbol{P}$ all we need is a monomorphism $G \to ev_P$ and a transformation $\Psi : G \to F$ with $x \in Im\Psi M$. Such a $\Psi : G \to F$ is called a partial P-cover of F that contains x.

Lemma 2.8. *An element $x \in FM$ has a support if and only if there exists a finite family $\{(a_i \in P_i)\}_{i=1}^n$ such that for every pair of arrows $h, k : M \to N$ we have that $hP_i(a_i) = kP_i(a_i)$ for every $i = 1, .., n$ implies that $Fh(x) = Fk(x)$.*

Proof. The only if part is clear. For the if part simply consider $(a_1, .., a_n) \in \prod_{i=1}^n MP_i \simeq M(\prod_{i=1}^n P_i)$ □

Proposition 2.9. *Given F in $\boldsymbol{UC}(\underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P}), \underline{\underline{\boldsymbol{Set}}})$, M in $\boldsymbol{Mod}(\boldsymbol{P})$ we have that every $x \in FM$ has a support.*

Proof. Suppose not. That is suppose that for every finite family $d = \{(a_i \in P_i)\}_{i=1}^n$ there exists a pair of arrows $h_d, k_d : M \to N_d$ in $\boldsymbol{Mod}(\boldsymbol{P})$ such that $h_d P_i(a_i) = k_d P_i(a_i)$ for every $i = 1, .., n$ but $Fh(x) \neq Fk(x)$. Let $\boldsymbol{D}$ be the set of finite families of the form $d = \{(a_i \in P_i)\}_{i=1}^n$ ordered by containment. For every d in $\boldsymbol{D}$ chose a pair of arrows $h_d, k_d : M \to N_d$ satisfying the property written above. Denote $\uparrow(d) = \{d' \in \boldsymbol{D} | d \subset d'\}$. Now, $M1 = 1$ and therefore $\boldsymbol{D}$ is nonempty, and for every $d, d' \in \boldsymbol{D}$ we have that $\uparrow(d) \cap \uparrow(d') = \uparrow(d \cup d')$. Therefore there exists an ultrafilter $\mathcal{U}$ on $\boldsymbol{D}$ such that for every $d \in \boldsymbol{D}$ we have $\uparrow(d) \in \mathcal{U}$. Consider the diagram

$$M \xrightarrow{\delta M} M^{\mathcal{U}} \underset{\prod_{\boldsymbol{D}} k_d/\mathcal{U}}{\overset{\prod_{\boldsymbol{D}} h_d/\mathcal{U}}{\rightrightarrows}} \prod_{\boldsymbol{D}} N_d/\mathcal{U}$$

where δ is the diagonal ultramorphism. Given $a \in MP$ consider $d = \{(a \in MP)\} \in \boldsymbol{D}$. Then for every $d' \in \uparrow(d)$ we have that $h_{d'}P(a) = k_{d'}P(a)$, therefore we have that $\langle h_{d'}P(a)\rangle_{d' \in \uparrow d} = \langle k_{d'}P(a)\rangle_{d' \in \uparrow d}$ in $\prod_{\boldsymbol{D}} N_d P/\mathcal{U}$. Therefore

$$\prod_{\boldsymbol{D}} h_d/\mathcal{U} \circ \delta M = \prod_{\boldsymbol{D}} k_d/\mathcal{U} \circ \delta M.$$

Consider the following diagram

$$\begin{tikzcd}
& F(M^{\mathcal{U}}) \arrow[r, shift left, "F(\prod_{\boldsymbol{D}} h_d/\mathcal{U})"] \arrow[r, shift right, "F(\prod_{\boldsymbol{D}} k_d/\mathcal{U})"'] \arrow[dd, "{[\mathcal{U},F]\langle M\rangle_{\boldsymbol{D}}}"] & F(\prod_{\boldsymbol{D}} N_d/\mathcal{U}) \arrow[dd, "{[\mathcal{U},F]\langle N_d\rangle_{\boldsymbol{D}}}"] \\
FM \arrow[ur, "F(\delta M)"] \arrow[dr, "\delta FM"'] & & \\
& (FM)^{\mathcal{U}} \arrow[r, shift left, "\prod_{\boldsymbol{D}} Fh_d/\mathcal{U}"] \arrow[r, shift right, "\prod_{\boldsymbol{D}} Fk_d/\mathcal{U}"'] & \prod_{\boldsymbol{D}} FN_d/\mathcal{U}
\end{tikzcd}$$

The left triangle commutes because F is an ultrafunctor and the right square clearly commutes sequentially. Therefore both compositions in

$$\begin{tikzcd}
FM \arrow[r, "\delta FM"] & (FM)^{\mathcal{U}} \arrow[r, shift left, "\prod_{\boldsymbol{D}} Fh_d/\mathcal{U}"] \arrow[r, shift right, "\prod_{\boldsymbol{D}} Fk_d/\mathcal{U}"'] & \prod_{\boldsymbol{D}} FN_d/\mathcal{U}
\end{tikzcd}$$

are equal. We then have that $\langle Fh_d(x)\rangle = \langle Fk_d(x)\rangle$ in $\prod_{\boldsymbol{D}} FN_d/\mathcal{U}$. Since we assumed that $Fh_d(x) \neq Fk_d(x)$ for every $d \in \boldsymbol{D}$ we have a contradiction. $\square$

For the next couple of propositions we use the notation from Proposition 1.20.

Lemma 2.10. *Given $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$, P in $\boldsymbol{P}$, $x \in FM$ and $a \in MP$, we have that $a \in MP$ is a support for x if and only if the only element of $\Theta(M,a)(1)$ is a support for $x \in F \circ (- \circ \Delta_P)(\Theta(M,a))$* $\square$

Proposition 2.11. *Let $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ be an ultrafunctor, P be an object of $\boldsymbol{P}$, M in $\boldsymbol{Mod}(\boldsymbol{P})$, $a \in MP$ and $x \in FM$. If there exist a subobject $r \rightarrowtail 1$ in $\boldsymbol{P}/P$ and an ultranatural transformation $\Phi : ev_r \to F \circ (- \circ \Delta_P)$ such that $\Theta(M,a)(r) = 1$ and $x \in Im\Phi\Theta(M,a)$ then there exists a subobject $Q \rightarrowtail P$ with $a \in MQ$ and an ultranatural transformation $\Psi : ev_Q \to F$ such that $\Psi M(a) = x$*

Proof. Consider a diagram

$$\begin{tikzcd}
Q \arrow[rr, tail, "m"] \arrow[dr, "r"'] & & P \arrow[dl, "1_P"] \\
& P &
\end{tikzcd}$$

in $\boldsymbol{P}/P$ and assume we have an ultranatural transformation $\Phi : ev_r \to F \circ (- \circ \Delta_P)$ satisfying the requirement of the proposition. By the definition of Θ it is clear that $a \in MQ$. Define $\Psi : ev_Q \to F$ as follows. Given N in $\boldsymbol{Mod}(\boldsymbol{P})$ and $b \in NQ$ we have $\Phi\Theta(N,b) : \Theta(N,b)(r) \to FN$. Since $b \in NQ$ we have that $\Theta(N,b)(r) = 1$. Define $\Psi N(b) = \Phi\Theta(N,b)(\bullet)$ (where $\bullet$ is the only element of $\Theta(N,b)(r)$). It is not hard to see that Ψ is an ultranatural transformation and that $\Psi Q(a) = x$. $\square$

The proposition above and the lemma preceding it tell us that when we have a support $a \in MP$ for $x \in FM$ it is enough to assume that $P = 1$ and that a is the only element of $M1$. Now, $\bullet \in M1$ is a support for $x \in FM$ if for every pair of morphisms $M \underset{k}{\overset{h}{\rightrightarrows}} N$ in $\boldsymbol{Mod}(\boldsymbol{P})$ we have that $Fh(x) = Fk(x)$

If $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ is a pre-ultrafunctor consider the category $\boldsymbol{Mod}^*(\boldsymbol{P}) = \boldsymbol{Mod}(\boldsymbol{P}) \coprod El(F)$, where $El(F)$ is the category of elemnts of F with forgetful functor $El(F) \to \boldsymbol{Mod}(\boldsymbol{P})$. If M is an object of $\boldsymbol{Mod}(\boldsymbol{P})$ we denote it by $(M,*)$ when we see it as an object in $\boldsymbol{Mod}^*(\boldsymbol{P})$, whereas an object (N,x) in $El(F)$ is also denoted by (N,x) when seen as an object of $\boldsymbol{Mod}^*(\boldsymbol{P})$. We say that (N,x) is a proper object if $x \neq *$, otherwise we say it is improper. We give $\boldsymbol{Mod}^*(\boldsymbol{P})$ a pre-ultracategory structure as follows. If $(I,\mathcal{U})$ is an ultrafilter and $\langle (M_i,x_i)\rangle_I$ is an I-family of objects of $\boldsymbol{Mod}^*(\boldsymbol{P})$, consider the set $J = \{i \in I | x_i \neq *\}$. Define

$$\prod (M_i,x_i)/\mathcal{U} = \begin{cases} (\prod M_i/\mathcal{U}, *) & \text{if } J \notin \mathcal{U} \\ (\prod M_i/\mathcal{U}, [\mathcal{U},F]\langle M_i\rangle^{-1}(\langle x_j\rangle_J)) & \text{if } J \in \mathcal{U} \end{cases}$$

and if $\langle f_i\rangle : \langle (M_i,x_i)\rangle \to \langle (N_i,y_i)\rangle$ is a morphism in $\boldsymbol{Mod}^*(\boldsymbol{P})^I$ then $\langle f_i\rangle \mapsto \prod f_i/\mathcal{U}$. We have a forgetful preultrafunctor $\boldsymbol{Mod}^*(\boldsymbol{P}) \to \boldsymbol{Mod}(\boldsymbol{P})$ such that $(M,x) \mapsto M$.

If we carry out the construction above with $id : \boldsymbol{Set} \to \boldsymbol{Set}$ instead of F we get a pre-ultracategory that we denote by $\boldsymbol{Set}^*$.

The preultrafunctor $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ induces a functor $F^* : \boldsymbol{Mod}^*(\boldsymbol{P}) \to \boldsymbol{Set}^*$ such that $F^*(M,x) = (FM,x)$ and $F^*h = Fh$ for every $h : (M,x) \to (N,y)$ in $\boldsymbol{Mod}^*(\boldsymbol{P})$. F^* turns into a pre-ultrafunctor if we define $[\mathcal{U},F^*]\langle (M_i,x_i)\rangle = [\mathcal{U},F]\langle M_i\rangle$ for every $\langle (M_i,x_i)\rangle$ in $\boldsymbol{Mod}^*(\boldsymbol{P})^I$.

Lemma 2.12. *Given a pre-ultrafunctor (ultrafunctor) $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ we have that subobjects of F in $\boldsymbol{PUC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$ ($\boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$) are in one to one* *correspondence with classes $\mathcal{C}$ of objects of* $\boldsymbol{Mod}^*(\boldsymbol{P})$ *that satisfy the conditions 0)-3) below*

0) For every M *in* $\boldsymbol{Mod}(\boldsymbol{P})$ *we have* $(M,*)\in\mathcal{C}$.

1) If $(M,x)\in\mathcal{C}$ *and* $f:(M,x)\to(N,y)$ *is a morphism in* $\boldsymbol{Mod}^*(\boldsymbol{P})$ *then* $(N,y)\in\mathcal{C}$.

2) For any ultrafilter $(I,\mathcal{U})$ *and any object* $\langle(M_i,x_i)\rangle$ *in* $\boldsymbol{Mod}^*(\boldsymbol{P})$ *with* $(M_i,x_i)\in\mathcal{C}$ *for every* $i\in I$ *we have that* $\prod(M_i,x_i)/\mathcal{U}\in\mathcal{C}$.

3) If $(I,\mathcal{U})$ *is an ultrafilter and* $\langle(M_i,x_i)\rangle$ *is an object of* $\boldsymbol{Mod}^*(\boldsymbol{P})^I$ *such that* $\prod(M_i,x_i)/\mathcal{U}\in\mathcal{C}$ *then there exists a set* $J\in\mathcal{U}$ *such that for every* $j\in J$, $(M_j,x_j)\in\mathcal{C}$.

Proof. Start with a subobject $G\overset{\mu}{\rightarrowtail}F$. Define the class

$$\mathcal{C}_G=\boldsymbol{Mod}(\boldsymbol{P})\coprod\{(M,x)\in El(F)|x\in Im\,\mu M\}.$$

Clearly $\mathcal{C}_G$ satisfies 0). If $(M,x)\in\mathcal{C}_G$ is proper and $f:(M,x)\to(N,y)$ in $\boldsymbol{Mod}^*(\boldsymbol{P})$ then, since $x\in Im\,\mu M$ and the diagram

$$\begin{tikzcd} GM \arrow[r, "Gf"] \arrow[d, "\mu M"'] & GN \arrow[d, "\mu N"] \\ FM \arrow[r, "Ff"'] & FN \end{tikzcd}$$

commutes, we have that $y\in Im\,\mu N$. If (M,x) is improper then $(N,y)=(N,*)\in\mathcal{C}_G$. Therefore $\mathcal{C}_G$ satisfies 1). Let $(I,\mathcal{U})$ be an ultrafilter and $\langle(M_i,x_i)\rangle$ be an object in $\boldsymbol{Mod}^*(\boldsymbol{P})^I$. Let $J=\{i\in I|x_i\neq *\}$. If $J\notin\mathcal{U}$ then clearly $\prod(M_i,x_i)/\mathcal{U}\in\mathcal{C}_G$. Assume then that $J\in\mathcal{U}$. Then for every $j\in J$ we have that $x_j\in Im\,\mu M_j$. Since μ is a pre-ultranatural transformation we have that the diagram

$$\begin{tikzcd} G(\prod M_i/\mathcal{U}) \arrow[r, "{[\mathcal{U},G]\langle M_i\rangle}"] \arrow[d, "\mu(\prod M_i/\mathcal{U})"'] & \prod GM_i/\mathcal{U} \arrow[d, "\prod \mu M_i/\mathcal{U}"] \\ F(\prod M_i/\mathcal{U}) \arrow[r, "{[\mathcal{U},F]\langle M_i\rangle}"'] & \prod FM_i/\mathcal{U} \end{tikzcd} \tag{2.1}$$

commutes. Then it is clear that $[\mathcal{U},F]\langle M_i\rangle^{-1}(\langle x_j\rangle_J)\in Im\,\mu\prod M_i/\mathcal{U}$, that is $\mathcal{C}_G$ satisfies 2). For 3) Assume that $\prod(M_i,x_i)/\mathcal{U}\in\mathcal{C}_G$. if $J=\{i\in I|x_i\neq *\}\notin\mathcal{U}$ then for every $i \in I - J$ we have that $(M_i, x_i) \in \mathcal{C}_G$. Suppose then that $J \in \mathcal{U}$. We have that $[\mathcal{U}, F]\langle M_i\rangle^{-1}(\langle x_j\rangle_J) \in Im\,\mu(\prod M_i/\mathcal{U})$. We then can find an element $\langle y_k\rangle_K \in \prod GM_i/\mathcal{U}$ such that $\mu(\prod M_i/\mathcal{U})([\mathcal{U}, G]\langle M_i\rangle^{-1}(\langle y_k\rangle_K)) = [\mathcal{U}, F]\langle M_i\rangle^{-1}(\langle x_j\rangle_J)$. This means that $\prod \mu M_i/\mathcal{U}(\langle y_k\rangle_K) = \langle x_j\rangle_J$. Therefore there exists a set $L \subset J \cap K$ with $L \in \mathcal{U}$ such that for every $\ell \in L$ we have $\mu M_\ell(y_\ell) = x_\ell$. That is for every $\ell \in L$ we have that $(M_\ell, x_\ell) \in \mathcal{C}_G$ so we have 3). It is easy to show that if the classes determined by two subobjects of F coincide then they are the same subobject.

Assume now that we have a class $\mathcal{C}$ of objects of $\boldsymbol{Mod}^*(\boldsymbol{P})$ that satisfies 0)-3) above. Define $G_\mathcal{C} : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ such that $G_\mathcal{C}(M) = \{x \in FM | (M, x) \in \mathcal{C}\}$.

If $h : M \to N$ is a morphism of models then condition 1) guarantees that $Fh : FM \to FN$ restricts

$$\begin{tikzcd} FM \arrow[r, "Fh"] & FN \\ G_\mathcal{C}M \arrow[u] \arrow[r, "G_\mathcal{C}h"'] & G_\mathcal{C}N \arrow[u] \end{tikzcd}$$

With these definitions we have that $G_\mathcal{C}$ is a subfunctor of F.

We want to define $[\mathcal{U}, G]\langle M_i\rangle_I : G_\mathcal{C}(\prod M_i/\mathcal{U}) \to \prod G_\mathcal{C}M_i/\mathcal{U}$ such that the diagram 2.1 commutes. Let $x \in G_\mathcal{C}(\prod M_i/\mathcal{U})$. We have then that $(\prod M_i/\mathcal{U}, x) \in \mathcal{C}$. Let $\langle x_j\rangle_J = [\mathcal{U}, F]\langle M_i\rangle_I(x)$. Then by 3) there exists $K \subset J, K \in \mathcal{U}$ such that for every $k \in K$, $(M_k, x_k) \in \mathcal{C}$. Therefore $\langle x_k\rangle_K \in \prod G_\mathcal{C}M_i/\mathcal{U}$. Define $[\mathcal{U}, G]\langle M_i\rangle(x) = \langle x_k\rangle_K$. Since $[\mathcal{U}, F]\langle M_i\rangle$ is an isomorphism it is easy to see that $[\mathcal{U}, G]\langle M_i\rangle$ is mono. Use 2) to show that $[\mathcal{U}, G]\langle M_i\rangle$ is onto. This gives us a subobject $G_\mathcal{C}$ of F in $\boldsymbol{PUC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$. It is easy to see that the association $\mathcal{C} \mapsto G_\mathcal{C}$, $G \mapsto \mathcal{C}_G$ between classes satisfying 0)-3) and subobjects of F in $\boldsymbol{PUC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$ are inverses. It is not hard to see that if F is an ultrafunctor then $G_\mathcal{C}$ is also an ultrafunctor. $\square$

Assume now that the only element of $M_0 1$ is a support for $x_0 \in FM_0$. A diagram of the form

$$\begin{tikzcd} G \arrow[r, tail] \arrow[d, "\Phi"'] & ev_1 \simeq 1 \\ F & \end{tikzcd}$$

is the same thing as a subobject $G \rightarrowtail ev_1 \times F \simeq F$ that satisfies $x, x' \in GM$ implies $x = x'$. That is, we need a class $\mathcal{C}$ satisfying 0)-3) above plus

4) $(M, x), (M, x') \in \mathcal{C}$ with $x, x' \in FM$ implies that $x = x'$.

We also want the class $\mathcal{C}$ to satisfy

5) $(M_0, x_0) \in \mathcal{C}$.

For the proof we will have to consider bigger and bigger small subcategories of the category $\boldsymbol{Mod}^*(\boldsymbol{P})$. Here is the definition of the small subcategories we will need.

Definition 2.7. Let $\boldsymbol{P}$ be a small pretopos and $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ be an ultrafunctor. A pair $(\boldsymbol{C}, \mathcal{S})$ is called a small approximation of $\boldsymbol{Mod}^*(\boldsymbol{P})$ provided that

i. $\boldsymbol{C}$ is a small subcategory of $\boldsymbol{Mod}^*(\boldsymbol{P})$

ii. $\mathcal{S}$ is a set of triples of the form $(I, \mathcal{U}, I \xrightarrow{g} Ob(\boldsymbol{C}))$ where $(I, \mathcal{U})$ is an ultrafilter.

iii. For every $(I, \mathcal{U}, g) \in \mathcal{S}$ the ultraproduct $\prod g(i)/\mathcal{U}$ is in $\boldsymbol{C}$.

iv. For every $g : \{0\} \to Ob(\boldsymbol{C})$ we have that $(\{0\}, \mathcal{U}_0, g) \in \mathcal{S}$ where $(\{0\}, \mathcal{U}_0)$ is the only possible ultrafilter over $\{0\}$.

v. If $(I, \mathcal{U}, g) \in \mathcal{S}$ and $g' : I \to Ob(\boldsymbol{C})$ is such that

$$I \underset{g'}{\overset{g}{\rightrightarrows}} Ob(\boldsymbol{C}) \xrightarrow{i} \boldsymbol{Mod}^*(\boldsymbol{P}) \xrightarrow{U} \boldsymbol{Mod}(\boldsymbol{P})$$

commutes then $(I, \mathcal{U}, g') \in \mathcal{S}$.

Let κ be the cardinality of $\boldsymbol{P}$ (that is $\kappa = \#(Ar(\boldsymbol{P}))$). We say that a small approximation $(\boldsymbol{C}, \mathcal{S})$ of $\boldsymbol{Mod}^*(\boldsymbol{P})$ is closed if it satisfies

vi. For every M in $\boldsymbol{Mod}(\boldsymbol{P})$ such that $\#M := \#(\coprod_{P \in \boldsymbol{P}} MP) \leq \kappa$ there exists $(N, *) \in \boldsymbol{C}$ such that $\#N \leq \kappa$ and $N \simeq M$.

vii. For every $(M, *), (N, *)$ in $\boldsymbol{C}$ such than $M \equiv N$ (elementary equivalent) there is an ultrafilter $(I, \mathcal{U})$ such that $(I, \mathcal{U}, g_1), (I, \mathcal{U}, g_2) \in \mathcal{S}$, with $g_1 : I \to Ob(\boldsymbol{C})$ is the constant map with value $(M, *)$, $g_2 : I \to Ob(\boldsymbol{C})$ is the constant map with value $(N, *)$ and $M^{\mathcal{U}} \simeq N^{\mathcal{U}}$.

Given a small approximation $(\boldsymbol{C}, \mathcal{S})$ of $\boldsymbol{Mod}^*(\boldsymbol{P})$ a $(\boldsymbol{C}, \mathcal{S})$-subobject of F is a family $\mathcal{C} \subset Ob(\boldsymbol{C})$ satisfying 0)-3) above when 2) and 3) are restricted to elements of $\mathcal{S}$.

A partial cover of F relative to $(\boldsymbol{C}, \mathcal{S})$ is a $(\boldsymbol{C}, \mathcal{S})$-subobject of F that satisfies 4).

Remark 2.1. Given a pair $(\boldsymbol{C},\mathcal{S})$ satisfying i-iii we can always find a pair $(\boldsymbol{C}',\mathcal{S}')$ satisfying i-v and such that $\boldsymbol{C}$ is a subcategory of $\boldsymbol{C}'$ and $\mathcal{S} \subset \mathcal{S}'$.

Remark 2.2. Given a small approximation $(\boldsymbol{C},\mathcal{S})$ we can always find a small close approximation $(\boldsymbol{C}',\mathcal{S}')$ such that $\boldsymbol{C}$ is a subcategory of $\boldsymbol{C}'$ and $\mathcal{S} \subset \mathcal{S}'$. This is a consequence of the Keisler-Shelah isomorphism theorem that says that given two models M, N such that $M \equiv N$ there exists an ultrafilter $(I,\mathcal{U})$ such that $M^{\mathcal{U}} \simeq N^{\mathcal{U}}$.

We now show that for every small approximation $(\boldsymbol{C},\mathcal{S})$ and any $x_0 \in FM_0$ with support the unique element of $M_0 1$ we can find a partial cover $\mathcal{C}$ of F relative to $(\boldsymbol{C},\mathcal{S})$ such that $\mathcal{C}$ satisfies 5). We start by putting (M_0, x_0) in $\mathcal{C}$. Notice that conditions 0)-2) can always be fulfilled by adding more and more objects to $\mathcal{C}$, however condition 3) involves the choice of a set in an ultrafilter. We will make all the necessary choices and repeat the process. In this way we can obtain a $\mathcal{C}$ that satisfies 0)-3) and 5) but not necessarily 4). We will assume that for all possible choices we obtain a family $\mathcal{C}$ that fails to fulfill 4) and we will get a contradiction. This process involves the recursive construction of an ultragraph.

So let $(\boldsymbol{C},\mathcal{S})$ be a small approximation of $\boldsymbol{Mod}^*(\boldsymbol{P})$ and assume that $\bullet \in M_0 1$ is a support for $x_0 \in FM_0$. Let $\kappa = \#\boldsymbol{C}$ and $\alpha_0 = \kappa^+$.

We construct the ultragraph $\boldsymbol{G}$ and the ultradiagram $D : \boldsymbol{G} \to \boldsymbol{Mod}^*(\boldsymbol{P})$ as follows.

For every $(M, *)$ in $\boldsymbol{C}$ we put a node φ_M. We also put a node φ_0. Define

$\boldsymbol{G}_0^f = \{\varphi_0\} \cup \{\varphi_M | (M, *) \text{ is in } \boldsymbol{C}\}$

$\boldsymbol{G}_0^b = \emptyset$

No edges in $\boldsymbol{G}_0$

$\Theta_0 = \emptyset$

$D_0 : \boldsymbol{G}_0 \to \boldsymbol{C}$ is such that $\varphi_0 \mapsto (M_0, x_0)$ and $\varphi_M \mapsto (M, *)$.

Let $0 < \alpha < \alpha_0$ and suppose we have made the corresponding definitions for all $\alpha' < \alpha$. Define

$\boldsymbol{G}_{<\alpha}^f = \bigcup_{\alpha'<\alpha} \boldsymbol{G}_{\alpha'}^f$

$\boldsymbol{G}_{<\alpha}^b = \bigcup_{\alpha'<\alpha} \boldsymbol{G}_{\alpha'}^b$

$\boldsymbol{G}_{<\alpha} = \bigcup_{\alpha'<\alpha} \boldsymbol{G}_{\alpha'}$

$\Theta_{<\alpha} = \bigcup_{\alpha'<\alpha} \Theta_{\alpha'}$

$D_{<\alpha} = \bigcup_{\alpha'<\alpha} D_{\alpha'}$

Let Θ_α be the set whose elements are of the form $\langle \alpha, I, \mathcal{U}, g; f; J, \mathcal{V}, g' \rangle$ such that

I. $(J, \mathcal{V}, g') \in \mathcal{S}$.

II. $g : I \to \boldsymbol{G}^f_{<\alpha}$.

III. $(I, \mathcal{U}, I \xrightarrow{g} \boldsymbol{G}^f_{<\alpha} \xrightarrow{D_{<\alpha}} \boldsymbol{C}) \in \mathcal{S}$.

IV. $I_0 = \{i \in I | D_{<\alpha}(g(i)) \text{ is proper}\} \in \mathcal{U}$

V. $f : \prod D_{<\alpha} g(i)/\mathcal{U} \to \prod g'(j)/\mathcal{V}$ is a morphism in $\boldsymbol{C}$.

Notice that condition IV implies that $\prod D_{<\alpha} g(i)/\mathcal{U}$ is a proper object.

For every $t = \langle \alpha, I_t, \mathcal{U}_t, g_t; f_t; J_t, \mathcal{V}_t, g'_t \rangle \in \Theta_\alpha$ take two nodes β_t, γ_t and for every $j \in J_t$ take a node (t, j). Define then

$\boldsymbol{G}^b_\alpha = \{\beta_t | t \in \Theta_\alpha\} \cup \{\gamma_t | t \in \Theta_\alpha\}$.

$\boldsymbol{G}^f_\alpha = \{(t, j) | t \in \Theta_\alpha \text{ and } j \in J_t\}$.

For every $t \in \Theta_\alpha$ put an edge $r_t : \beta_t \to \gamma_t$ in $\boldsymbol{G}_\alpha$.

$D_\alpha(\beta_t) = \prod D_{<\alpha} g_t(i)/\mathcal{U}_t$.

$D_\alpha(\gamma_t) = \prod g'_t(j)/\mathcal{V}_t$.

$D_\alpha(t, j) = g'_t(j)$.

$D_\alpha(r_t) = f$.

Finally define $\boldsymbol{G} = \boldsymbol{G}_{<\alpha_0}$ and $D = D_{<\alpha_0}$. We have that $\boldsymbol{G}$ is an ultragraph and D is an ultradiagram. Notice as well that D factors through $\boldsymbol{C}$.

Next we make formal the concept of possible choices of elements of ultrafilters for the family to satisfy 3).

Let Θ be a subset of Θ_{α_0}, and $\vec{A} = \langle A_t \rangle_{t \in \Theta}$ be a Θ-indexed family of sets such that $A_t \in \mathcal{V}_t$ for every $t \in \Theta$. We define recursively what it means for $t \in \Theta_\alpha$ and γ node of $\boldsymbol{G}$ to be $\vec{A}$-accessible.

First, φ_0 is $\vec{A}$-accessible.

For every M, φ_M is not $\vec{A}$-accessible.

Suppose we know what it means to be $\vec{A}$-accessible for $t \in \Theta_{<\alpha}$ and $\gamma \in \boldsymbol{G}_{<\alpha}$ for $0 < \alpha < \alpha_0$. Then

$t \in \Theta_\alpha$ is $\vec{A}$-accessible if and only if $\{i \in I_t | g_t(i) \text{ is } \vec{A}\text{-accessible}\} \in \mathcal{U}_t$.

β_t is $\vec{A}$-accessible if and only if t is $\vec{A}$-accessible.

γ_t is $\vec{A}$-accessible if and only if t is $\vec{A}$-accessible.

(t, j) is $\vec{A}$-accessible if and only if t is $\vec{A}$-accessible, $t \in \Theta$ and $j \in A_t$.

We say that $\vec{A} = \langle A_t \rangle_\Theta$ is regular if and only if for every $t \in \Theta_{<\alpha}$ we have $t \in \Theta$ if and only if t is $\vec{A}$-accessible. Define $\boldsymbol{G}(\vec{A}) = \{\gamma \in \boldsymbol{G} | \gamma \text{ is } \vec{A}\text{-accessible}\}$, and let $\mathcal{A} = \{\vec{A} | \vec{A} \text{ is regular}\}$.

Notice that $\mathcal{A}$ is a meet semilattice. Given $\vec{A} = \langle A_t \rangle_\Theta$ and $\vec{B} = \langle B_t \rangle_{\Theta'}$ construct $\vec{C} = \langle C_t \rangle_{\Theta''}$ recursively as follows. Suppose we know already what $\Theta'' \cap \Theta_{<\alpha}$ is and that we have already defined C_t for every $t \in \Theta'' \cap \Theta_{<\alpha}$. Then $t \in \Theta'' \cap \Theta_\alpha$ if and only if t is $\langle C_t : t \in \Theta'' \cap \Theta_{<\alpha} \rangle$-accessible and define $C_t = A_t \cap B_t \in \mathcal{V}_t$. $\vec{C} = \langle C_t \rangle_{\Theta''}$ is regular and $\vec{C} = \vec{A} \wedge \vec{B}$ in $\mathcal{A}$.

Lemma 2.13. *Given an ultradiagram $E : \boldsymbol{G} \to \boldsymbol{Set}$ and $a \in E(\varphi_0)$ there is an ultradiagram $E^* : \boldsymbol{G} \to \boldsymbol{Set}^*$ such that $E^*(\varphi_0) = (E(\varphi_0), a)$ and the diagram*

$$\begin{tikzcd} \boldsymbol{G} \arrow[rr, "E^*"] \arrow[dr, "E"'] & & \boldsymbol{Set}^* \arrow[dl, "U"] \\ & \boldsymbol{Set} & \end{tikzcd}$$

commutes, where U is the forgetful functor.

Proof. Define $E^*(\varphi_0) = (E(\varphi_0), a)$ and $E^*(\varphi_M) = (E(\varphi_M), *)$. Assume that $E^*(\gamma)$ has been defined for $\gamma \in \boldsymbol{G}_{<\alpha}$. Let $t \in \Theta_\alpha$, define $E^*(\beta_t) = \prod_{I_t} E(g_t(i))/\mathcal{U}_t$. Define $E^*(\gamma_t) = (E(\gamma_t), b)$ if b makes $E(r_t)$ a morphism $E^*(\beta_t) \to (E(\gamma_t), b)$ in $\boldsymbol{Set}^*$ (notice that there is a unique b with this property). Define $E^*(r_t) = E(r_t)$. Choose $J \in \mathcal{V}_t$ and $a_j \in E(t, j)$ for every $j \in J$ such that $b = \langle a_j \rangle_J$. Define $E^*(t, j) = (E(t, j), a_j)$ if $j \in J$ and $E^*(t, j) = (E(t, j), *)$ if $j \notin J$ $\square$

Lemma 2.14. *Given $\vec{A}$ regular the family $\mathcal{C} = \{(M, *) \in \boldsymbol{C}\} \cup \{D(\gamma) | \gamma \in \boldsymbol{G}(\vec{A})\}$ satisfies conditions 0)-3) and 5), where $D : \boldsymbol{G} \to \boldsymbol{Mod}^*(\boldsymbol{P})$ is the ultradiagram defined above.*

Proof. Clearly 0) is satisfied. Since $\varphi_0 \in \boldsymbol{G}(\vec{A})$ and $D(\varphi_0) = (M_0, x_0)$, $\mathcal{C}$ satisfies 5).

Assume $(M, x) \in \mathcal{C}$ is proper. We show that there exists $\gamma \in \boldsymbol{G}^f \cap \boldsymbol{G}(\vec{A})$ such that $D(\gamma) = (M, x)$. If $(M, x) = D(\beta_t)$ with t $\vec{A}$-accessible, $t \in \Theta_\alpha$, $\alpha < \alpha_0$ then $\{i \in I_t | g_t(i) \text{ is } \vec{A}\text{-accessible}\} \in \mathcal{U}_t$. Let $t' = \langle \alpha, I_t, \mathcal{U}_t, g_t; id_M; \{0\}, \mathcal{U}_0, g' \rangle$ where $g'(0) =$

(M,x). Then $t' \in \Theta_\alpha$, t' is $\vec{A}$-accessible and we have $D(t',0) = (M,x)$. Clearly $(t',0) \in \boldsymbol{G}^f \cap \boldsymbol{G}(\vec{A})$. The case $D(\gamma_t) = (M,x)$ is similar.

$\mathcal{C}$ satisfies 1): Let $(M,x) = D(\gamma)$ with $\gamma \in \boldsymbol{G}^f \cap \boldsymbol{G}(\vec{A})$ and $h : (M,x) \to (N,y)$ in $\boldsymbol{C}$. Suppose $\gamma \in \boldsymbol{G}^f_{<\alpha}$ with $\alpha < \alpha_0$. Let $t = \langle \alpha; \{0\}, \mathcal{U}_0, g; h; \{0\}, \mathcal{U}_0, g' \rangle$ where $g(0) = \gamma$ and $g'(0) = (N,y)$. Then $t \in \Theta_\alpha$. Since γ is $\vec{A}$-accessible we have that t is $\vec{A}$-accessible, this means that β_t and γ_t are also $\vec{A}$-accessible. Clearly $D(\gamma_t) = (N,y)$. That is $(N,y) \in \mathcal{C}$.

$\mathcal{C}$ satisfies 2): Let $(I,\mathcal{U},g) \in \mathcal{S}$ and with $g(i) = (M_i,x_i) \in \mathcal{C}$. If $J = \{i \in I | g(i)$ is proper$\} \notin \mathcal{U}$ then clearly $\prod g(i)/\mathcal{U} \in \mathcal{C}$. Assume then that $J \in \mathcal{U}$. For every $j \in J$ let $\gamma_j \in \boldsymbol{G}^f_{\alpha_j} \cap \boldsymbol{G}(\vec{A})$ such than $(M_j,x_j) = D(\gamma_j)$. Assume furthermore that $(M_j,x_j) = (M_{j'},x_{j'})$ implies $\gamma_j = \gamma_{j'}$ for $j,j' \in J$. Since the cardinality of $\{\alpha_j\} < \kappa$ there exists $\alpha < \alpha_0 = \kappa^+$ such that $\alpha_j < \alpha$ for every $j \in J$. Let $t = \langle \alpha; I, \mathcal{U}, g; id; \{0\}, \mathcal{U}_0, g' \rangle$ where $g(i) = \gamma(i)$ if $i \in J$, $g(i) = \varphi_{M_i}$ and $g'(0) = \prod D(g(i))/\mathcal{U}$. Notice that $\prod D(g(i))/\mathcal{U} = \prod (M_i,x_i)/\mathcal{U}$. Now, $t \in \Theta_\alpha$ and for every $j \in J, \gamma_j$ is $\vec{A}$-accessible, therefore t and β_t are $\vec{A}$-accessible. We have $\prod (M_i,x_i) = D(\beta_t)$.

$\mathcal{C}$ satisfies 3): Let (M_i,x_i) in $\boldsymbol{C}$ for $i \in I$ and assume $\prod (M_i,x_i)/\mathcal{U} \in \mathcal{C}$ with $(I,\mathcal{U},\langle (M_i,x_i) \rangle) \in \mathcal{S}$. If $\prod (M_i,x_i)/\mathcal{U} \in \mathcal{C}$ is improper then the conclusion is clear, so assume it is proper. Assume $\prod (M_i,x_i)/\mathcal{U} \in \mathcal{C} = D(\gamma)$ with $\gamma \in \boldsymbol{G}^f_\alpha \cap \boldsymbol{G}(\vec{A})$ and $\alpha < \alpha_0$. Let $t = \langle \alpha, \{0\}, \mathcal{U}_0, g; id; I, \mathcal{U}, \langle (M_i,x_i) \rangle \rangle \in \Theta_\alpha$ with $g(0) = \gamma$. Since γ is $\vec{A}$-accessible we have that t is $\vec{A}$-accessible. Since $\vec{A} = \langle A_{t'} \rangle_{t' \in \Theta}$ is regular we have that $t \in \Theta$. Then (t,j) is $\vec{A}$-accessible for every $j \in A_t$ and $D(t,j) = (M_j,x_j)$ for $j \in A_t$. $\square$

Lemma 2.15. *Given an ultrafunctor $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}, (\boldsymbol{C},\mathcal{S})$ a small approximation of $\boldsymbol{Mod}^*(\boldsymbol{P})$ and $x_0 \in FM_0$ with support the only element of $M_0 1$. There exists a partial cover $\mathcal{C}$ of F relative to $(\boldsymbol{C},\mathcal{S})$ such that $(M_0,x_0) \in \mathcal{C}$.*

Proof. Consider the ultradiagram $D : \boldsymbol{G} \to \boldsymbol{Mod}^*(\boldsymbol{P})$ defined above. We have seen that for $\vec{A}$ regular the family $\mathcal{C}_{\vec{A}} = \{(M,*) | (M,*)$ in $\boldsymbol{C}\} \cup \{D(\gamma) | \gamma \in \boldsymbol{G}(\vec{A})\}$ satisfies 0)-3) and 5). If for some regular $\vec{A}$ the family $\mathcal{C}_{\vec{A}}$ also satisfies 4) we are done. So let's assume that for every $\vec{A} \in \mathcal{A} = \{\vec{B} | \vec{B}$ is regular $\}$ the family $\mathcal{C}_{\vec{A}}$ does not satisfy 4). Then for every $\vec{A} \in \mathcal{A}$ we can find nodes $\gamma_1(\vec{A}), \gamma_2(\vec{A}) \in \boldsymbol{G}^f \cap \boldsymbol{G}(\vec{A})$ such that $D(\gamma_1(\vec{A})) = (M_{\vec{A}}, x_{\vec{A}1})$ and $D(\gamma_2(\vec{A})) = (M_{\vec{A}}, x_{\vec{A}2})$ are proper and $x_{\vec{A}1} \neq x_{\vec{A}2}$.

$\gamma_1(\vec{A}), \gamma_2(\vec{A})$ can be chosen in $\boldsymbol{G}^f \cap \boldsymbol{G}(\vec{A})$ as a consequence of the proof of the previous lemma). We know that $\mathcal{A}$ is a meet semilattice, so there exists an ultrafilter $\mathcal{W}$ on $\mathcal{A}$ such that for every $\vec{A} \in \mathcal{A}$, $\downarrow(\vec{A}) \in \mathcal{W}$. We construct a new ultragraph $\boldsymbol{G}_1$ as follows. $\boldsymbol{G}_1$ is obtained from $\boldsymbol{G}$ by adding a new bound node ℓ and assigning to it the triple $(\mathcal{A}, \mathcal{W}, g)$ where $g(\vec{A}) = \gamma_1(\vec{A})$. We define an ultramorphism $\delta_1 : ev_{\varphi_0} \to ev_\ell : \boldsymbol{UD}(\boldsymbol{G}, \boldsymbol{Set}) \to \boldsymbol{Set}$ as follows. Given an ultradiagram $E : \boldsymbol{G}_1 \to \boldsymbol{Set}$ consider the ultradiagram $E' = E|_{\boldsymbol{G}} : \boldsymbol{G} \to \boldsymbol{Set}$ and notice that E' essentially determines E. We can assume $E(\ell) = \prod E'(\gamma_1(\vec{A}))/\mathcal{W}$. Let $a \in E(\varphi_0)$, construct $E' : \boldsymbol{G} \to \boldsymbol{Set}^*$ as in lemma 2.13. If $E'^*(\gamma_1(\vec{A})) = (E(\gamma_1), a_1(\vec{A}))$ define $\delta_1 E(a) = \langle a_1(\vec{A})\rangle_{\mathcal{A}}$ in $\prod E(\gamma_1(\vec{A}))/\mathcal{W}$. It is not hard to see that $\delta_1 E(a) \neq *$, that it does not depend on the choice of E'^* and that δ_1 defines an ultramorphism.

Similarly, using $\gamma_2(\vec{A})$ instead of $\gamma_1(\vec{A})$ we obtain an ultragraph $\boldsymbol{G}_2$ and an ultramorphism $\delta_2 : ev_{\varphi_0} \to ev_\ell : \boldsymbol{UD}(\boldsymbol{G}_2, \boldsymbol{Set}) \to \boldsymbol{Set}$.

Consider the ultradiagram $\boldsymbol{G} \xrightarrow{D} \boldsymbol{C} \xrightarrow{U} \boldsymbol{Mod}(\boldsymbol{P})$ where D was defined above and U is the forgetful functor. We can extend DU to ultradiagrams

$$D_1 : \boldsymbol{G}_1 \to \boldsymbol{Mod}(\boldsymbol{P}) \qquad D_2 : \boldsymbol{G}_2 \to \boldsymbol{Mod}(\boldsymbol{P})$$

such that $D_1(\ell) = D_2(\ell) = \prod M_{\vec{A}}/\mathcal{W}$ and $D_1|_{\boldsymbol{G}} = D_2|_{\boldsymbol{G}} = DU$. Since δ_1, δ_2 are ultramorphisms over $\boldsymbol{Set}$ we have the corresponding ultramorphisms $\widehat{\delta}_1, \widehat{\delta}_2$ over $\boldsymbol{Mod}(\boldsymbol{P})$. We obtain a pair of homomorphisms

$$D_1(\varphi_0) = D_2(\varphi_0) = M_0 \overset{\widehat{\delta}_1 D_1}{\underset{\widehat{\delta}_2 D_2}{\rightrightarrows}} \prod M_{\vec{A}}/\mathcal{W} = D_1(\ell) = D_2(\ell)$$

Applying F we have

$$FM_0 \overset{F(\widehat{\delta}_1 D_1)}{\underset{F(\widehat{\delta}_2 D_2)}{\rightrightarrows}} F(\prod M_{\vec{A}}/\mathcal{W})$$

Since $x_0 \in FM_0$ has support $\bullet \in M_0 1$ we have

$$F(\widehat{\delta}_1 D_1)(x_0) = F(\widehat{\delta}_2 D_2)(x_0). \tag{2.2}$$

We show that $[\mathcal{W}, F]\langle M_{\vec{A}}\rangle(F(\widehat{\delta}_1 D_1)(x_0)) = [\langle x_{\vec{A}1}\rangle]$: Since F is an ultrafunctor we have that the diagram

$$\begin{tikzcd}
FM_0 \arrow[rr, "F(\widehat{\delta}_1 D_1)"] \arrow[dr, "\delta_1 FD_1"'] & & F(\prod M_{\vec{A}}/\mathcal{W}) \arrow[dl, "{[\mathcal{W},F]\langle M_{\vec{A}}\rangle}"] \\
& (\prod FM_{\vec{A}}/\mathcal{W}) &
\end{tikzcd}$$

commutes. So what we want to show is that $\delta_1 FD_1(x_0) = [\langle x_{\vec{A}1}\rangle]$. According to the definition of δ_1 we need a lifting of FD_1. Define $D^* : \boldsymbol{G}_1 \to \boldsymbol{Mod}^*(\boldsymbol{P})$ such that $D^*|_G = D$ and $D^*(\ell) = (\prod M_{\vec{A}}/\mathcal{W}, [\mathcal{W},F]\langle M_{\vec{A}}\rangle^{-1}([\langle x_{\vec{A}1}\rangle]))$. It is clear that the diagram

$$\begin{tikzcd}
\boldsymbol{G}_1 \arrow[r, "D^*"] \arrow[dr, "FD_1"'] & \boldsymbol{Mod}^*(\boldsymbol{P}) \arrow[r, "F^*"] & \boldsymbol{Set}^* \arrow[dl, "U"] \\
& \boldsymbol{Set} &
\end{tikzcd}$$

commutes, where $F^*(M,x) = (FM,x)$. We conclude that $\delta_1 FD_1(x_0) = [\langle x_{\vec{A}1}\rangle]$.

Similarly we can show that $\delta_2 FD_2(x_0) = [\langle x_{\vec{A}2}\rangle]$. By the way we chose $x_{\vec{A}1}$ and $x_{\vec{A}2}$ that $[\langle x_{\vec{A}1}\rangle] \neq [\langle x_{\vec{A}2}\rangle]$. This is in contradiction with 2.2. $\square$

Lemma 2.16. *Let $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ be an ultrafunctor, $(\boldsymbol{C},\mathcal{S})$ be a small closed approximation of $\boldsymbol{Mod}^*(\boldsymbol{P})$ and $\mathcal{C},\mathcal{D}$ be two $(\boldsymbol{C},\mathcal{S})$-subobjects of F. If for all $(M,*)$ in $\boldsymbol{C}$ with $\#M \leq \kappa$ and ever $x \in FM$ we have $(M,x) \in \boldsymbol{C}$ if and only if $(M,x) \in \mathcal{D}$, then $\mathcal{C} = \mathcal{D}$.*

Proof. Let $(N,*)$ be an object of $\boldsymbol{C}$. Since $(\boldsymbol{C},\mathcal{S})$ is a closed approximation we can find $(M,*)$ in $\boldsymbol{C}$ with $\#M \leq \kappa$ and $M \equiv N$ together with an ultrafilter $(I,\mathcal{U})$ with the following properties. There is an isomorphism $h : M^{\mathcal{U}} \to N^{\mathcal{U}}$ and $(I,\mathcal{U},g_1),(I,\mathcal{U},g_2) \in \mathcal{S}$ where $g_1, g_2 : I \to Ob(\boldsymbol{C})$ are constant functions with values $(M,*)$ and $(N,*)$ respectively. Consider the following diagram

$$\begin{tikzcd}
& (FM)^{\mathcal{U}} & (FN)^{\mathcal{U}} & \\
FM \arrow[ur, "\delta FM"] \arrow[dr, "F(\delta M)"'] & & & FN \arrow[ul, "\delta FN"'] \arrow[dl, "F(\delta N)"] \\
& F(M^{\mathcal{U}}) \arrow[uu, "{[\mathcal{U},F]\langle M\rangle}"'] \arrow[r, "Fh"] & F(N^{\mathcal{U}}) \arrow[uu, "{[\mathcal{U},F]\langle N\rangle}"] &
\end{tikzcd}$$

where δ denotes the diagonal. Notice that since F is an ultrafunctor the above diagram commutes.

Let $y \in FN$. We show that $(N, y) \in \mathcal{C}$ if and only if there exist $J \in \mathcal{U}$ and an $(M, x_j) \in \mathcal{C}$ for every $j \in J$ such that $F(\delta N)(y) = Fh([\mathcal{U}, F]\langle M\rangle^{-1}([\langle x_j\rangle]))$.

Assume first that $(N, y) \in \mathcal{C}$. Since $\mathcal{C}$ is a $(\boldsymbol{C}, \mathcal{S})$-subobject we have $\prod(N, y)/\mathcal{U} \in \mathcal{C}$. Let $z \in F(M^{\mathcal{U}})$ such that $Fh(z) = F(\delta N)(y)$. Then $h^{-1} : (N^{\mathcal{U}}, F(\delta N)(y)) \to (M^{\mathcal{U}}, z)$ is in $\boldsymbol{C}$. Therefore $(M^{\mathcal{U}}, z) \in \boldsymbol{C}$. Since $\mathcal{C}$ is a $(\boldsymbol{C}, \mathcal{S})$-subobject we can find a $J \in \mathcal{U}$ and objects $(M, x_j) \in \boldsymbol{C}$ for every $j \in J$ such that $[\mathcal{U}, F]\langle M\rangle^{-1}[\langle x_j\rangle] = z$. Now apply Fh. Conversely, assume that $F(\delta N)(y) = Fh([\mathcal{U}, F]\langle M\rangle^{-1}([\langle x_j\rangle_J]))$ for some $J \in \mathcal{U}$ and $(M, x_j) \in \mathcal{C}$ for every $j \in J$. Then $(M^{\mathcal{U}}, [\mathcal{U}, F]\langle M\rangle^{-1}([\langle x_j\rangle_J])) \in \mathcal{C}$. Since $h : ((M^{\mathcal{U}}, [\mathcal{U}, F]\langle M\rangle^{-1}([\langle x_j\rangle_J]))) \to (N, F(\delta N)(y))$ is in $\boldsymbol{C}$ we have that $(N, F(\delta N)(y)) \in \mathcal{C}$. This means that $(N, y) \in \mathcal{C}$.

We clearly have the same result for $\mathcal{D}$. Therefore $\mathcal{C} = \mathcal{D}$. $\square$

Lemma 2.17. *Let $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ be an ultrafunctor, M_0 a model in $\boldsymbol{Mod}(\boldsymbol{P})$ and $x_0 \in FM_0$. Assume that $\bullet \in M_0 1$ is a support for $x_0 \in FM_0$. Then there is a diagram of the form*

$$\begin{tikzcd} G \arrow[r, mapsto] \arrow[d, "\Phi"'] & ev_1 \simeq 1 \\ F & \end{tikzcd} \tag{2.3}$$

in $\boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$ such that $x_0 \in Im\,\Phi M_0$.

Proof. For every ordinal α give a small closed approximation $(\boldsymbol{C}_\alpha, \mathcal{S}_\alpha)$ such that

- If $\alpha < \beta$ then $\boldsymbol{C}_\alpha \subset \boldsymbol{C}_\beta$ and $\mathcal{S}_\alpha \subset \mathcal{S}_\beta$.
- $\bigcup_\alpha \boldsymbol{C}_\alpha = \boldsymbol{Mod}^*(\boldsymbol{P})$.
- $\bigcup_\alpha \mathcal{S}_\alpha$ is the set (in the second universe) of all the triples $(I, \mathcal{U}, g)$ with $(I, \mathcal{U})$ an ultrafilter and $g : I \to Ob(\boldsymbol{Mod}^*(\boldsymbol{P}))$.

It is not hard to see that such a sequence of small closed approximations exists. Since $(\boldsymbol{C}_0, \mathcal{S}_0)$ is a small close approximation we can find a small set Λ and a family of models $\{M_\ell\}_{\ell \in \Lambda}$ such that

- $\#M_\ell \leq \kappa$ for every $\ell \in \Lambda$.
- $(M_\ell, *)$ is an object in $\boldsymbol{C}_0$ for every $\ell \in \Lambda$.

- For every model M in $\boldsymbol{Mod}(\boldsymbol{P})$ with $\#M \leq \kappa$ there is an $\ell \in \Lambda$ such that $M \simeq M_\ell$.

For every ordinal α let $\mathcal{C}_\alpha$ be a partial cover of F relative to $(\boldsymbol{C}_\alpha, \mathcal{S}_\alpha)$ with $(M_0, x_0) \in \mathcal{C}_\alpha$. For every ordinal α and every $\ell \in \Lambda$ define $X_{\alpha\ell} = \{x \in FM_\ell | (M_\ell, x) \in \mathcal{C}_\alpha\}$. Every α determines the family $\langle X_{\alpha\ell}\rangle$. Notice that since Λ is small and F is fixed there is a small set of such families. It follows that there is a family $\langle X_\ell \rangle$ such that the set (in the second universe) $\Xi = \{\alpha | \alpha$ is an ordinal and $\langle X_{\alpha\ell}\rangle = \langle X_\ell\rangle\}$ is unbounded. If $\alpha, \beta \in \Xi$ with $\alpha < \beta$ then by lemma 2.16 we have that $\mathcal{C}_\alpha = \mathcal{C}_\alpha \cap \mathcal{C}_\beta$, that is, $\mathcal{C}_\alpha \subset \mathcal{C}_\beta$. Define $\mathcal{C} = \bigcup_{\alpha \in \Xi} \mathcal{C}_\alpha$. By the remarks after the proof of lemma 2.12 $\mathcal{C}$ corresponds to a diagram of the form 2.3 above. □

By proposition 2.11 we have

Corollary 2.18. *Let $F : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ be an ultrafunctor, M_0 in $\boldsymbol{Mod}(\boldsymbol{P})$, $x_0 \in FM_0$, P in $\boldsymbol{P}$ and $a \in M_0P$ such that a is a support for x_0. There is a diagram of the form*

$$\begin{tikzcd} G \arrow[r, tail] \arrow[d, "\Phi"'] & ev_P \\ F & \end{tikzcd}$$

in $\boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$ such that $a \in GM$ and $\Phi M_0(a) = x_0$. □

In a result similar to 2.16 we show that an ultranatural transformation is determined by its values at models of size at most $\kappa = \#\boldsymbol{P}$

Lemma 2.19. *If $\Phi, \Psi : F \to G : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ are ultra-natural transformations between ultrafunctors such that for every model M in $\boldsymbol{Mod}(\boldsymbol{P})$ of cardinality $\#M \leq \kappa$ we have $\Phi M = \Psi M$ then $\Phi = \Psi$*

Proof. Let N be a model. Choose a model M of cardinality at most κ, an ultrafilter $(I, \mathcal{U})$ and an isomorphism $h : M^{\mathcal{U}} \to N^{\mathcal{U}}$. Let $y \in FN$. Since h is an isomorphism there exists $z \in F(M^{\mathcal{U}})$ such that $Fh(z) = F\delta N(y)$. Let $J \in \mathcal{U}$ and $x_j \in FM$ for every $j \in J$ such that $[\mathcal{U}, F]\langle M\rangle(z) = [\langle x_j\rangle_J]$. Since Φ is an ultranatural transformation the diagram

$$\begin{tikzcd} F(M^{\mathcal{U}}) \arrow[r, "{[\mathcal{U},F]\langle M\rangle}"] \arrow[d, "\Phi(M^{\mathcal{U}})"'] & (FM)^{\mathcal{U}} \arrow[d, "(\Phi M)^{\mathcal{U}}"] \\ G(M^{\mathcal{U}}) \arrow[r, "{[\mathcal{U},G]\langle M\rangle}"'] & (GM)^{\mathcal{U}} \end{tikzcd}$$

commutes. It follows that $\Phi(M^{\mathcal{U}})(z) = [\mathcal{U},G]\langle M\rangle^{-1}[\langle \Phi M(x_j)\rangle]$. Using the naturality of Φ applied to h we conclude that $\Phi(N^{\mathcal{U}})(F\delta N(y)) = Gh([\mathcal{U},G]\langle M\rangle^{-1}[\langle \Phi M(x_j)\rangle])$. Using the commutativity of

$$\begin{tikzcd} FN \arrow[r, "F\delta N"] \arrow[d, "\Phi N"'] & F(N^{\mathcal{U}}) \arrow[d, "\Phi(N^{\mathcal{U}})"] \\ GN \arrow[r, "G\delta N"'] & G(N^{\mathcal{U}}) \end{tikzcd}$$

we have $G\delta N(\Phi N(y)) = Gh([\mathcal{U},G]\langle M\rangle^{-1}[\langle \Phi M(x_j)\rangle])$. The same reasoning shows that $G\delta N(\Psi N(y)) = Gh([\mathcal{U},G]\langle M\rangle^{-1}[\langle \Psi M(x_j)\rangle])$. Since $\#M \leq \kappa$ we have that $\Phi M(x_j) = \Psi M(x_j)$ for every $j \in J$. The result follows from this. □

Proposition 2.20. *If $\boldsymbol{P}$ is a small pretopos, then every F in $\boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$ has a finite cover via $ev : \boldsymbol{P} \to \boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$.*

Proof. Since $\boldsymbol{P}$ is small there is a small set of ultrafunctors of the form ev_P with P in $\boldsymbol{P}$. According to Lemma 2.19 an ultrafunctor $ev_P \to F$ is determined by its values on models of size at most κ. From lemma 2.6 we know that $ev_P : \boldsymbol{P} \to \boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$ is subobject full. It follow that there is a small set $\mathcal{T}$ of diagrams of the form $F \xleftarrow{\Phi} G \rightarrowtail ev_P$ such that for any diagram $F \xleftarrow{\Phi} G' \rightarrowtail ev_P$ there is a diagram $(F \xleftarrow{\Phi} G \rightarrowtail ev_P) \in \mathcal{T}$ and an isomorphism $G \to G'$ such that the diagram

$$\begin{tikzcd} F & G \arrow[l, "\Phi"'] \arrow[r] \arrow[d] & ev_P \\ & G' \arrow[ul, "\Phi'"] \arrow[ur] & \end{tikzcd}$$

commutes.

For every model M in $\boldsymbol{Mod}(\boldsymbol{P})$ and $x \in FM$ we know that there is a diagram of the form $(F \xleftarrow{\Phi} G \rightarrowtail ev_P)$ with $x \in Im\,\Phi M$. By what we said above we may assume that $(F \xleftarrow{\Phi} G \rightarrowtail ev_P) \in \mathcal{T}$.

Let $\mathcal{P}_\omega(\mathcal{T})$ denote the set of finite subsets of $\mathcal{T}$ ordered by inclusion. Assume that for every $T \in \mathcal{T}$, $T = \{F \xleftarrow{\Phi_i} G_i \rightarrowtail ev_{P_i}\}_{i=1}^n$ there are a model M_T and $x_T \in FM_T$ such that $x_T \notin \bigcup_{i=1}^n \Phi_i M_T$. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{P}_\omega(\mathcal{T})$ such that for every $T \in \mathcal{T}$ we have that $\uparrow(T) \in \mathcal{U}$. Consider $[\mathcal{U}, F]\langle M_T\rangle^{-1}[\langle x_T\rangle] \in F(\prod M_T/\mathcal{U})$. We can find $(F \xleftarrow{\Phi} G \rightarrowtail ev_P) \in \mathcal{T}$ such that $[\mathcal{U}, F]\langle M_T\rangle^{-1}[\langle x_T\rangle \in Im\,\Phi \prod M_T/\mathcal{U}$. This means that there is $J \in \mathcal{U}$ such that for every $T \in J$, $x_T \in Im\,\Phi M_T$. If $T \in \uparrow \{F \xleftarrow{\Phi} G \rightarrowtail ev_P)\} \cap J \in \mathcal{U}$ then we have that $x_T \in Im\,\Phi M_T$. On the other hand, since $(F \xleftarrow{\Phi} G \rightarrowtail ev_P) \in T$ we have $x_T \notin Im\,\Phi M_T$. A contradiction. There exists then $T \in \boldsymbol{P}_\omega(\mathcal{T})$ such that for every model M and every $x \in FM$ there is an element $(F \xleftarrow{\Phi} G \rightarrowtail ev_P) \in T$ with $x \in Im\,\Phi M$. T is then a finite cover of F via $ev : \boldsymbol{P} \to \boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$. □

We have shown that for a small pretopos $\boldsymbol{P}$ the functor

$$ev : \boldsymbol{P} \to \boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$$

is conservative (Proposition 2.7), subobject full (Proposition 2.6) and that every F in $\boldsymbol{UC}(\boldsymbol{Mod}(\boldsymbol{P}), \boldsymbol{Set})$ has a finite cover via ev (Proposition 2.20). This is enough to prove Makkai's Theorem (Theorem 2.3).

Chapter 3

Continuous Families of Models

In this chapter we are going to consider categories of models of pretoposes as categories indexed over $\boldsymbol{Top}$, the category of topological spaces and continuous functions. Before we go into the definitions we want to give some motivation for taking this approach.

Given a continuous function $f : Y \to X$ in $\boldsymbol{Top}$ we obtain a geometric morphism $Sh(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} Sh(Y)$. Now, f^* preserves finite limits and all colimits, this in particular means that $f^* : Sh(X) \to Sh(Y)$ is an elementary functor. For any pretopos $\boldsymbol{P}$ composition with f^* induces a functor $\boldsymbol{Mod}_{Sh(X)}(\boldsymbol{P}) \to \boldsymbol{Mod}_{Sh(Y)}(\boldsymbol{P})$ which we also call f^*. We want to relate this with the ultraproduct functors (see 1.4). Let I be a set and consider it as a topological space with discrete topology, let βI be its Stone-Čech compactification and $\xi I : I \to \beta I$ be the usual embbeding. $\beta I = \{\mathcal{U} | \mathcal{U}$ is an ultrafilter on $I\}$, and a basis for the topology on βI is given by sets of the form $J^* = \{\mathcal{U} \in \beta I | J \in \mathcal{U}\}$ for subsets $J \subset I$. We will show later that $\xi I_* : Sh(I) \to Sh(\beta I)$ is an elementary functor (see Proposition 3.18). We have an equivalence of categories given by $P : \boldsymbol{Set}^I \to Sh(I)$ where $P\langle A_i \rangle(J) = \prod_{j \in J} A_j$ and $P\langle f_i \rangle(J) = \prod_{j \in J} f_j : \prod_{j \in J} A_j \to \prod_{j \in J} B_j$ for every $J \subset I$ and $\langle f_i \rangle : \langle A_i \rangle \to \langle B_i \rangle$ in $\boldsymbol{Set}^I$. If $\mathcal{U}$ is an ultrafilter on I then we have a function $1 \xrightarrow{\mathcal{U}} \beta I$ that sends the only element of 1 to $\mathcal{U}$.

Lemma 3.1. *The composition* $\boldsymbol{Set}^I \xrightarrow{P} Sh(I) \xrightarrow{\xi I_*} Sh(\beta I) \xrightarrow{\mathcal{U}^*} \boldsymbol{Set}$ *is naturally isomorphic to the ultraproduct functor defined by* $\mathcal{U}$.

Proof. Denote by $L : Sh(\beta I) \to LH/\beta I$ the usual equivalence where $LH/\beta I$ is the category of local homeomorphisms over βI. If we start with a family $\langle A_i \rangle_{i \in I}$ in $\mathbf{Set}^I$ we have that

$$L(\xi I_*(P\langle A_i \rangle_{i\in I})) = \coprod_{\mathcal{F}\in\beta I} \varinjlim_{\substack{W \text{ open} \\ \mathcal{F}\in W}} \xi I_*(P\langle A_i \rangle_{i\in I})(W)$$

using the fact that the sets of the form J^* form a basis for the topology of βI we have

$$\begin{aligned} L(\xi I_*(P\langle A_i \rangle_{i\in I})) &\simeq \coprod_{\mathcal{F}\in\beta I} \varinjlim_{\substack{J\subset I \\ \mathcal{F}\in J^*}} \xi I_*(P\langle A_i \rangle_{i\in I})(J^*) \\ &= \coprod_{\mathcal{F}\in\beta I} \varinjlim_{\substack{J\subset I \\ J\in\mathcal{F}}} (P\langle A_i \rangle_{i\in I})(\xi I^{-1}(J^*)) \\ &= \coprod_{\mathcal{F}\in\beta I} \varinjlim_{J\in\mathcal{F}} P\langle A_i \rangle_{i\in I}(J) \\ &= \coprod_{\mathcal{F}\in\beta I} \varinjlim_{J\in\mathcal{F}} \prod_{i\in J} A_i \end{aligned}$$

Therefore, the fiber over $\mathcal{U}$ is $\varinjlim_{J\in\mathcal{U}} \prod_{i\in J} A_i$. We proceed similarly with families of morphisms. $\square$

Assuming we know that $\xi I_* : Sh(I) \to Sh(\beta I)$ is elementary (see 3.18 below) we have that composition with ξI_* induces a functor $\mathbf{Mod}_{Sh(I)}(\mathbf{P}) \to \mathbf{Mod}_{Sh(\beta I)}(\mathbf{P})$ (called ξI_* as well) for any pretopos $\mathbf{P}$. We have an equivalence $F : \mathbf{Mod}(\mathbf{P})^I \to \mathbf{Mod}_{Sh(I)}(\mathbf{P})$ given by $F\langle M_i \rangle(P) = \langle M_i P \rangle$ and $F\langle \tau_i \rangle(P) = \langle \tau_i P \rangle$ for every P in $\mathbf{P}$ and every $\langle \tau_i \rangle : \langle M_i \rangle \to \langle N_i \rangle$ in $\mathbf{Mod}(\mathbf{P})^I$.

Corollary 3.2. *The composition*

$$\mathbf{Mod}(\mathbf{P})^I \xrightarrow{F} \mathbf{Mod}_{Sh(I)}(\mathbf{P}) \xrightarrow{\xi I_*} \mathbf{Mod}_{Sh(\beta I)}(\mathbf{P}) \xrightarrow{\mathcal{U}^*} \mathbf{Mod}(\mathbf{P})$$

is naturally isomorphic to the ultraproduct functor defined by $\mathcal{U}$. $\square$

We obtain then the ultraproduct functors from continuous functions in $\mathbf{Top}$.

3.1 Indexed Category Theory

Basic Definitions

We review indexed category theory, as in [19]; in [3] the approach is via fibrations. To start with, we need a category $\mathbf{T}$ with finite limits, that we call the base category.

We further assume that $\boldsymbol{T}$ is locally small.

Definition 3.1. A $\boldsymbol{T}$-indexed category $\mathcal{A}$ consists of the following data

1. A category $\mathcal{A}^X$ for every object X in $\boldsymbol{T}$.
2. A functor $f^* : \mathcal{A}^X \to \mathcal{A}^Y$ for every arrow $Y \xrightarrow{f} X$ in $\boldsymbol{T}$.
3. A natural isomorphism

$$\begin{tikzcd} \mathcal{A}^X \arrow[r, bend left, "(1_X)^*"{name=U}] \arrow[r, bend right, "1_{\mathcal{A}^X}"'{name=D}] & \mathcal{A}^X \arrow[from=U, to=D, Rightarrow, "\simeq"] \end{tikzcd}$$

for every X in $\boldsymbol{T}$.

4. A natural isomorphism

$$\begin{tikzcd} \mathcal{A}^X \arrow[rr, "f^*"] \arrow[dr, "(f \circ g)^*"'] & \arrow[d, phantom, "\xrightarrow{\simeq}"] & \mathcal{A}^Y \arrow[dl, "g^*"] \\ & \mathcal{A}^Z & \end{tikzcd}$$

for every $Z \xrightarrow{g} Y \xrightarrow{f} X$ in $\boldsymbol{T}$.

Subject to the following coherence axioms

A1. The diagrams

$$\begin{tikzcd} (f \circ 1_Y)^* \arrow[r, "\simeq"] \arrow[d, "1"'] & (1_Y)^* \circ f^* \arrow[d, "\simeq"] \\ f^* \arrow[r, "1"'] & 1_{\mathcal{A}^Y} \circ f^* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (1_X \circ f)^* \arrow[r, "\simeq"] \arrow[d, "1"'] & f^* \circ (1_X)^* \arrow[d, "\simeq"] \\ f^* \arrow[r, "1"'] & f^* \circ 1_{\mathcal{A}^X} \end{tikzcd}$$

commute for every $Y \xrightarrow{f} X$ in $\boldsymbol{T}$.

A2. The diagram

$$\begin{tikzcd} (f \circ g \circ h)^* \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & h^* \circ (f \circ g)^* \arrow[d, "\simeq"] \\ (g \circ h)^* \circ f^* \arrow[r, "\simeq"'] & h^* \circ g^* \circ f^* \end{tikzcd}$$

commutes for every $W \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$ in $\boldsymbol{T}$.

Definition 3.2. Given $\boldsymbol{T}$-indexed categories $\mathcal{A}$ and $\mathcal{B}$, a $\boldsymbol{T}$-indexed functor $F : \mathcal{A} \to \mathcal{B}$ consists of the following data:

1. A functor $F^X : \mathcal{A}^X \to \mathcal{B}^X$ for every X in $\boldsymbol{T}$.

2. A natural isomorphism

$$\begin{tikzcd} \mathcal{A}^X \arrow[r, "f^*"] \arrow[d, "F^X"'] & \mathcal{A}^Y \arrow[d, "F^Y"] \arrow[dl, phantom, "\swarrow \simeq"] \\ \mathcal{B}^X \arrow[r, "f^*"'] & \mathcal{B}^Y \end{tikzcd}$$

for every $Y \xrightarrow{f} X$ in $\boldsymbol{T}$.

Subject to the following coherence axioms:

B1. The diagram

$$\begin{tikzcd} F^X \circ (1_X)^* \arrow[r] \arrow[dd] & F^X \circ 1_{\mathcal{A}^X} \arrow[dr, "1"] & \\ & & F^X \\ (1_X)^* \circ F^X \arrow[r] & 1_{\mathcal{B}^X} \circ F^X \arrow[ur, "1"'] & \end{tikzcd}$$

commutes for every X in $\boldsymbol{T}$.

B2. The diagram

$$\begin{tikzcd} F^Z \circ (f \circ g)^* \arrow[r] \arrow[d] & F^Z \circ g^* \circ f^* \arrow[r] & g^* \circ F^Y \circ f^* \arrow[d] \\ (f \circ g)^* \circ F^X \arrow[rr] & & g^* \circ f^* \circ F^X \end{tikzcd}$$

commutes for every $Z \xrightarrow{g} Y \xrightarrow{f} X$ in $\boldsymbol{T}$.

Composition of $\boldsymbol{T}$-indexed functors is defined in the obvious way.

Definition 3.3. Given $\boldsymbol{T}$-indexed functors $F, G : \mathcal{A} \to \mathcal{B}$, a $\boldsymbol{T}$-indexed natural transformation $\tau : F \to G$ consists of a natural transformation $\tau^X : F^X \to G^X$ for every X in $\boldsymbol{T}$, such that the diagram

$$\begin{tikzcd} F^Y \circ f^* \arrow[r, "\tau^Y f^*"] \arrow[d, "\simeq"'] & G^Y \circ f^* \arrow[d, "\simeq"] \\ f^* \circ F^X \arrow[r, "f^* \tau^X"'] & f^* \circ G^X \end{tikzcd}$$

commutes for every $Y \xrightarrow{f} X$ in $\boldsymbol{T}$.

$\boldsymbol{T}$-indexed natural transformations also compose in the obvious way.

Examples

We will be interested in the case where $\boldsymbol{T}$ is the category $\boldsymbol{Top}$ of topological spaces. As an example we have the $\boldsymbol{Top}$-indexed category $\mathcal{SET}$. Given a topological space we define $\mathcal{SET}^X$ to be the category $Sh(X)$ of sheaves over X. If $f : Y \to X$ is a continuous function then $f^* : \mathcal{SET}^X \to \mathcal{SET}^Y$ is the usual $f^* : Sh(X) \to Sh(Y)$.

Here is another example. If $\mathcal{A}$ is a $\boldsymbol{T}$-indexed category and, $\boldsymbol{C}$ is a small (ordinary) category then we define the $\boldsymbol{T}$-indexed category $[\boldsymbol{C}, \mathcal{A}]$ as follows; $[\boldsymbol{C}, \mathcal{A}]^X = (\mathcal{A}^X)^{\boldsymbol{C}}$ for X in $\boldsymbol{T}$. If $Y \xrightarrow{f} X$ is an arrow of $\boldsymbol{T}$, then $f^* : [\boldsymbol{C}, \mathcal{A}]^X \to [\boldsymbol{C}, \mathcal{A}]^Y$ is such that $(\boldsymbol{C} \xrightarrow{H} \mathcal{A}^X) \mapsto (\boldsymbol{C} \xrightarrow{H} \mathcal{A}^X \xrightarrow{f^*} \mathcal{A}^Y)$

If $\mathcal{A}$ is a $\boldsymbol{T}$-indexed category, we define the $\boldsymbol{T}$-indexed category $\mathcal{A}^{op}$, such that $(\mathcal{A}^{op})^X = (\mathcal{A}^X)^{op}$ and for $Y \xrightarrow{f} X$ in $\boldsymbol{T}$, the transition functor is $(f^*)^{op}$. If $\mathcal{B}$ is another $\boldsymbol{T}$-indexed category, we can define the $\boldsymbol{T}$-indexed category $\mathcal{A} \times \mathcal{B}$ such that $(\mathcal{A} \times \mathcal{B})^X = \mathcal{A}^X \times \mathcal{B}^X$ and the functor corresponding to f is $f^* \times f^* : \mathcal{A}^X \times \mathcal{B}^X \to \mathcal{A}^Y \times \mathcal{B}^Y$.

$\boldsymbol{T}$ itself can be regarded as a $\boldsymbol{T}$-indexed category $\mathcal{T}$ in the following way; Define $\mathcal{T}^X = \boldsymbol{T}/X$ for X in $\boldsymbol{T}$ and, for $Y \xrightarrow{f} X$ define f^* to be the pullback functor along f.

Small Homs

Questions of size concerning a $\boldsymbol{T}$-indexed category should be considered with respect to the base category. Given A and A' in $\mathcal{A}^X$, we have the functor

$$H_{A,A'} : (\boldsymbol{T}/X)^{op} \to \boldsymbol{SET},$$

such that for every

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

in $\mathcal{T}/X$, we have $H_{A,A'}(f) = \mathcal{A}^Y(f^*A, f^*A')$, and

$$H_{A,A'}(h) : \mathcal{A}^Y(f^*A, f^*A') \to \mathcal{A}^Z(g^*A, g^*A')$$

is such that

$$(f^*A \xrightarrow{a} f^*A') \mapsto (g^*A = (fh)^*A \xrightarrow{\simeq} h^*f^*A \xrightarrow{h^*a} h^*f^*A' \xrightarrow{\simeq} (fh)^*A' = g^*A').$$

Definition 3.4. A $\boldsymbol{T}$-indexed category $\mathcal{A}$ is said to have small homs if for every X in $\boldsymbol{T}$, A, A' in $\mathcal{A}^X$ there exists an object $hom^X(A, A') : Hom^X(A, A') \to X$ in $\boldsymbol{T}/X$ and a natural isomorphism

$$\boldsymbol{T}/X(\_\,, hom^X(A, A')) \to H_{A,A'}.$$

We say that $\mathcal{A}$ has small homs at 1 if the above condition is satisfied for $X = 1$

Whenever we have such an isomorphism we represent it by a horizontal line as follows

$$\begin{array}{cl} f^*A \to f^*A' & \text{in } \mathcal{A}^Y \\ \hline f \to hom^X(A, A') & \text{in } \boldsymbol{T}/X. \end{array}$$

Suppose that $\mathcal{A}$ has small homs. A morphism $(b, b') : (A, A') \to (B, B')$ in $(\mathcal{A}^X)^{op} \times \mathcal{A}^X$ induces a natural transformation $H_{b,b'} : H_{A,A'} \to H_{B,B'}$ in the obvious way. This corresponds to a natural transformation

$$\boldsymbol{T}/X(\_\,, hom^X(A, A')) \to \boldsymbol{T}/X(\_\,, hom^X(B, B')).$$

By Yoneda, this last transformation is represented by a unique morphism in $\boldsymbol{T}/X$ that we denote by $hom^X(b, b') : hom^X(A, A') \to hom^X(B, B')$. If we have $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ in $\boldsymbol{T}$, then

$$\begin{array}{cl} g \to f^*hom^X(A, A') & \textit{in } \boldsymbol{T}/Y \\ \hline fg \to hom^X(A, A') & \textit{in } \boldsymbol{T}/X \\ \hline (fg)^*A \to (fg)^*A' & \textit{in } \mathcal{A}^Z \\ \hline g^*f^*A \to g^*f^*A' & \textit{in } \mathcal{A}^Z \\ \hline g \to hom^Y(f^*A, f^*A') & \textit{in } \boldsymbol{T}/Y. \end{array}$$

This means that $hom^Y(f^*A, f^*A') \simeq f^*hom^X(A, A')$ in $\boldsymbol{T}/Y$. Therefore, if we define $hom(\_, \_) : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{T}$ such that for every X in $\boldsymbol{T}$, $hom(\_, \_)^X(A, A') = hom^X(A, A')$ and $hom(\_, \_)^X(b, b') = hom^X(b, b')$ we obtain

Lemma 3.3. *If the $\boldsymbol{T}$-indexed category $\mathcal{A}$ has small homs then $hom(\_, \_) : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{T}$ is a $\boldsymbol{T}$-indexed functor.* $\square$

3.1.1 Stability

Definition 3.5. We say that a $\boldsymbol{T}$-indexed category $\mathcal{A}$ has $\boldsymbol{T}$-stable colimits if for every X in $\boldsymbol{T}$, $\mathcal{A}^X$ has colimits and for every $f : Y \to X$ the functor $f^* : \mathcal{A}^X \to \mathcal{A}^Y$ preserves colimits.

Similarly we define the concepts of $\boldsymbol{T}$-stable coproducts, $\boldsymbol{T}$-stable finite limits etc. This concept of $\boldsymbol{T}$-stability should not be confused with the somewhat related concept of stability under pullbacks. To avoid confusion we will use the word *universal* to mean stable under pullback in this section.

A related concept is

Definition 3.6. Given a $\boldsymbol{T}$-indexed category $\mathcal{A}$, an object X in $\boldsymbol{T}$ and a monomorphism $m : A_0 \rightarrowtail A$ in $\mathcal{A}^X$, we say that m is $\boldsymbol{T}$-stable if for every $Y \xrightarrow{f} X$ in $\boldsymbol{T}$ we have that f^*m is a monomorphism in $\mathcal{A}^Y$. We say that $\mathcal{A}$ has $\boldsymbol{T}$-stable monomorphisms if every monomorphism in $\mathcal{A}^X$ is $\boldsymbol{T}$-stable for every X in $\boldsymbol{T}$. We say that a subobject $m : A_0 \rightarrowtail A$ in $\mathcal{A}^X$ is $\boldsymbol{T}$-stable if m is a $\boldsymbol{T}$-stable monomorphism.

3.1.2 Well Powered Categories

Given a $\boldsymbol{T}$-indexed category $\mathcal{A}$ and A in $\mathcal{A}^X$, define the functor

$$Ssub((\_)^*A) : (\boldsymbol{T}/X)^{op} \to \boldsymbol{SET}$$

such that for every

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

in $\boldsymbol{T}/X$, $Ssub((\_)^*A)(f) = Ssub(f^*A)$ is the set of $\boldsymbol{T}$-stable subobjects of f^*A, and $Ssub((\_)^*A)(h) : Ssub(g^*A) \to Ssub(f^*A)$ is $(B \rightarrowtail f^*A) \mapsto (h^*B \rightarrowtail h^*f^*A \xrightarrow{\simeq} g^*A)$, for every $\boldsymbol{T}$-stable subobject $B \rightarrowtail f^*A$.

Definition 3.7. A $\boldsymbol{T}$-indexed category $\mathcal{A}$ is said to be well powered if for every X in $\boldsymbol{T}$, A in $\mathcal{A}^X$, there exists an object $sub^X(A) : Sub^X(A) \to X$ in $\boldsymbol{T}/X$, and a natural isomorphism $\boldsymbol{T}/X(\_, sub^X(A)) \to Ssub((\_)A)$. We say that $\mathcal{A}$ is well powered at 1 if the above condition is satisfied for $X = 1$.

If the $\boldsymbol{T}$-indexed category $\mathcal{A}$ has $\boldsymbol{T}$-stable pullbacks and is well powered, then for every $a : A \to A'$ in $\mathcal{A}^X$ we can define the natural transformation $Ssub((\_)^*a)$: $Ssub((\_)^*A') \to Ssub((\_)^*A)$ such that for any $Y \xrightarrow{f} X$ in $\boldsymbol{T}/X$ we have that $Ssub(f^*a)(B \rightarrowtail f^*A')$ is the pullback

$$\begin{tikzcd}
Ssub(f^*a)(B \rightarrowtail f^*A') \arrow[r] \arrow[d, tail] & B \arrow[d, tail] \\
f^*A \arrow[r, "f^*a"'] & f^*A'.
\end{tikzcd}$$

This induces a natural transformation $\boldsymbol{T}/X(\_, sub^X(A')) \to \boldsymbol{T}/X(\_, sub^X(A))$. By Yoneda this last natural transformation is represented by a morphism in $\boldsymbol{T}/X$ that we denote by $sub^X(a) : sub^X(A') \to sub^X(A)$.

Define $sub(\_) : \mathcal{A}^{op} \to \mathcal{T}$ such that $sub(\_)^X(A) = sub^X(A)$, and $sub(\_)^X(a) = sub^X(a)$, for every $X \in \boldsymbol{T}$ and $A \xrightarrow{a} A'$ in $\mathcal{A}^X$. As for *hom* we have

Lemma 3.4. *If the $\boldsymbol{T}$-indexed category $\mathcal{A}$ has $\boldsymbol{T}$-stable pullbacks and is well powered then $sub(\_) : \mathcal{A}^{op} \to \mathcal{T}$ is a $\boldsymbol{T}$-indexed functor.* □

Notice that if $\mathcal{A}$ has $\boldsymbol{T}$-stable pullbacks then every monomorphism is $\boldsymbol{T}$-stable.

3.1.3 Adjoint Functors

Definition 3.8. If $F : \mathcal{A} \to \mathcal{B}$ is a $\boldsymbol{T}$-indexed functor, we say that F has a right adjoint if there exists a $\boldsymbol{T}$-indexed functor $R : \mathcal{B} \to \mathcal{A}$ and $\boldsymbol{T}$-indexed natural transformations $\eta : 1_F \to RF$ and $\epsilon : FR \to 1_R$ such that the diagrams

$$\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FRF \arrow[d, "\epsilon F"] \\
 & F
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
RFR \arrow[r, "R\epsilon"] & R \\
R \arrow[u, "\eta R"] \arrow[ur, "1_R"'] &
\end{tikzcd}$$

commute.

3.1.4 Internal Functors

Let $\mathbb{D}$ be the $\boldsymbol{T}$-category

$$\begin{tikzcd}
D_2 \arrow[r, shift left=2, "\pi_0"] \arrow[r, "\gamma"] \arrow[r, shift right=2, "\pi_1"'] & D_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & D_0, \arrow[l, "id"']
\end{tikzcd}$$

that is, $\mathbb{D}$ is a category object in $\boldsymbol{T}$.

Definition 3.9. Let $\mathcal{A}$ be a $\boldsymbol{T}$-indexed category, and $\mathbb{D}$ a $\boldsymbol{T}$-category as above. An internal functor from $\mathbb{D}$ to $\mathcal{A}$ is a pair $(A, \delta_0^*A \xrightarrow{\xi} \delta_1^*A)$ with A in $\mathcal{A}^{D_0}$ and ξ a morphism in $\mathcal{A}^{D_1}$, such that the diagrams

$$\begin{tikzcd}
A \arrow[rr, "\simeq"] \arrow[dr, "\simeq"'] & & id^*\delta_0^*A \arrow[dl, "id^*\xi"] \\
& id^*\delta_1^*A &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\pi_0^*\delta_0^*A \arrow[r, "\pi_0^*\xi"] \arrow[d, "\simeq"] & \pi_0^*\delta_1^*A \arrow[r, "\simeq"] & \pi_1^*\delta_0^*A \arrow[d, "\pi_1^*\xi"] \\
\gamma^*\delta_0^*A \arrow[r, "\gamma^*\xi"'] & \gamma^*\delta_1^*A \arrow[r, "\simeq"'] & \pi_1^*\delta_1^*A
\end{tikzcd}$$

commute. Given another internal functor $(B, \delta_0^*B \xrightarrow{\chi} \delta_1^*B)$ from $\mathbb{D}$ to $\mathcal{A}$, an internal natural transformation $a : (A, \delta_0^*A \xrightarrow{\xi} \delta_1^*A) \to (B, \delta_0^*B \xrightarrow{\chi} \delta_1^*B)$ is a morphism $a : A \to B$ in $\mathcal{A}^{D_0}$ such that the diagram

$$\begin{tikzcd}
\delta_0^*A \arrow[r, "\xi"] \arrow[d, "\delta_0^*a"'] & \delta_1^*A \arrow[d, "\delta_1^*a"] \\
\delta_0^*B \arrow[r, "\chi"'] & \delta_1^*B
\end{tikzcd}$$

commutes.

Internal natural transformations compose in the obvious way, and we obtain the category $\mathcal{A}^{\mathbb{D}}$ whose objects are internal functors from $\mathbb{D}$ to $\mathcal{A}$ and whose morphisms are internal natural transformations. Furthermore, we can $\boldsymbol{T}$-index $\mathcal{A}^{\mathbb{D}}$ as follows. Given an object X in $\boldsymbol{T}$, form the $\boldsymbol{T}$-category $\mathbb{D} \times X$ and define $(\mathcal{A}^{\mathbb{D}})^X = \mathcal{A}^{\mathbb{D}\times X}$. If $f : X \to Y$ is a morphism in $\boldsymbol{T}$, then $f^* : \mathcal{A}^{\mathbb{D}\times X} \to \mathcal{A}^{\mathbb{D}\times Y}$ is such that $(C, \delta_0^*C \xrightarrow{\mu} \delta_1^*C) \mapsto ((D_0 \times f)^*C, (D_1 \times f)^*\mu)$.

If $H : \mathbb{D} \to \mathbb{C}$ is the $\boldsymbol{T}$-functor

$$\begin{tikzcd}[column sep=large, row sep=large]
D_2 \arrow[r, shift left=2, "\pi_0"] \arrow[r, "\gamma"] \arrow[r, shift right=2, "\pi_1"'] \arrow[d, "H_2"'] & D_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] \arrow[d, "H_1"'] & D_0 \arrow[l, "id"'] \arrow[d, "H_0"] \\
C_2 \arrow[r, shift left=2, "\pi_0"] \arrow[r, "\gamma"] \arrow[r, shift right=2, "\pi_1"'] & C_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] & C_0 \arrow[l, "id"']
\end{tikzcd}$$

between $\boldsymbol{T}$-categories, we define $H^* : \mathcal{A}^{\mathbb{C}} \to \mathcal{A}^{\mathbb{D}}$ such that $(A, \delta_0^* A \xrightarrow{\xi} \delta_1^* A) \mapsto (H_0^* A, \delta_0^* H_0^* A \xrightarrow{\simeq} H_1^* \delta_0^* A \xrightarrow{H_1^*(\xi)} H_1^* \delta_1^* A) \xrightarrow{\simeq} \delta_1^* H_0^* A)$.

If $F : \mathcal{A} \to \mathcal{B}$ is a $\boldsymbol{T}$-indexed functor between $\boldsymbol{T}$-indexed categories, we can induce the functor $F^{\mathbb{D}} : \mathcal{A}^{\mathbb{D}} \to \mathcal{B}^{\mathbb{D}}$ such that $(A, \delta_0^* A \xrightarrow{\xi} \delta_1^* A) \mapsto (F^{D_0} A, \delta_0^* F^{D_0} A \xrightarrow{\simeq} F^{D_1} \delta_0^* \xrightarrow{F^{D_1}\xi} F^{D_1} \delta_1^* A \xrightarrow{\simeq} \delta_1^* F^{D_0} A)$. It is not hard to see that when $H : \mathbb{D} \to \mathbb{C}$ as above we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}^{\mathbb{C}} \arrow[r, "H^*"] \arrow[d, "F^{\mathbb{C}}"'] & \mathcal{A}^{\mathbb{D}} \arrow[d, "F^{\mathbb{D}}"] \\
\mathcal{B}^{\mathbb{C}} \arrow[r, "H^*"'] & \mathcal{B}^{\mathbb{D}}
\end{tikzcd}$$

Small Limits

We can define a $\boldsymbol{T}$-indexed functor $\Delta_{\mathbb{D}} : \mathcal{A} \to \mathcal{A}^{\mathbb{D}}$ such that for every X in $\boldsymbol{T}$ and $a : A \to A'$ in $\mathcal{A}^X$, $\Delta_{\mathbb{D}}^X(A) = (\pi_X^* A, (\delta_0 \times X)^* \pi_X^* A \xrightarrow{\simeq} (\delta_1 \times X)^* \pi_X^* A)$, and $\Delta_{\mathbb{D}}^X(a) = \pi_X^* a$, where $\pi_X : D_0 \times X \to X$ is the projection.

Definition 3.10. We say that the $\boldsymbol{T}$-indexed category $\mathcal{A}$ has $\mathbb{D}$-limits if the $\boldsymbol{T}$-indexed functor $\Delta_{\mathbb{D}}$ has right adjoint $\underleftarrow{lim}_{\mathbb{D}}$.

$\mathbb{D}$-colimits are defined in the same fashion, requiring a left adjoint instead of a right adjoint.

3.2 Functor Categories

We consider now categories of the form $\boldsymbol{T}\text{-}ind(\mathcal{A},\mathcal{B})$ of $\boldsymbol{T}$-indexed functors form $\mathcal{A}$ to $\mathcal{B}$. As in ordinary category theory $\boldsymbol{T}\text{-}ind(\mathcal{A},\mathcal{B})$ inherits its properties from $\mathcal{B}$.

Proposition 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\boldsymbol{T}$-indexed categories. If $\mathcal{B}$ has $\boldsymbol{T}$-stable limits then the category $\boldsymbol{T}\text{-}ind(\mathcal{A},\mathcal{B})$ has limits and if $F : \mathcal{A} \to \mathcal{C}$ is a $\boldsymbol{T}$-indexed functor then the functor $\boldsymbol{T}\text{-}ind(F,\mathcal{B}) : \boldsymbol{T}\text{-}ind(\mathcal{C},\mathcal{B}) \to \boldsymbol{T}\text{-}ind(\mathcal{A},\mathcal{B})$ preserves limits.*

Proof. Let $\Gamma : \boldsymbol{I} \to \boldsymbol{T}\text{-}ind(\mathcal{A},\mathcal{B})$ be a diagram. For every X in $\boldsymbol{T}$ we obtain a diagram $\Gamma^X : \boldsymbol{I} \to \boldsymbol{CAT}(\mathcal{A}^X,\mathcal{B}^X)$ such that $\Gamma^X I = (\Gamma I)^X$ and $\Gamma^X i = (\Gamma i)^X$ for every $i : I \to I'$ in $\boldsymbol{I}$. Define $\Theta^X = \varprojlim_I \Gamma^X I$. Since $\mathcal{B}^X$ has limits we have that for every A in $\mathcal{A}^X$, $\Theta^X(A) = \varprojlim_I (\Gamma I^X(A))$. Given $f : Y \to X$ we obtain a natural isomorphism

$$\Theta^Y f^* = \varprojlim_I \Gamma I^Y f^* \xrightarrow{\simeq} \varprojlim_I f^* \Gamma I^X \xrightarrow{\simeq} f^* \varprojlim_I \Gamma I^X = f^* \Theta^X$$

where the first arrow is induced by the isomorphisms $\Gamma I^Y f^* \xrightarrow{\simeq} f^* \Gamma I^X$ and the second isomorphism by the fact that f^* preserves limits. It is not hard to see that these isomorphisms satisfy coherence, making $\Theta : \mathcal{A} \to \mathcal{B}$ a $\boldsymbol{T}$-indexed functor. For every I in $\boldsymbol{I}$ we define $\pi_I^X : \Theta^X \to \Gamma I^X$ as the projection. It is easy to see that this definition makes π_I a $\boldsymbol{T}$-indexed functor and the family $\langle \Theta \xrightarrow{\pi_i} \Gamma I \rangle$ a cone. The universal property is clear. $\square$

Remark 3.1. Notice that the above proposition remains true if we replace limits by finite limits or coproducts etc, provided they are $\boldsymbol{T}$-stable in $\mathcal{B}$. Notice furthermore that the limits (or colimits, etc) are calculated doubly pointwise, that is they are calculated as the limit in $\boldsymbol{T}\text{-}ind(\mathcal{A}^X,\mathcal{B}^X)$ and they are pointwise at every $\boldsymbol{T}\text{-}ind(\mathcal{A}^X,\mathcal{B}^X)$.

Lemma 3.6. *If $\mathcal{B}$ has $\boldsymbol{T}$-stable strict initial object then $\boldsymbol{T}\text{-}ind(\mathcal{A},\mathcal{B})$ has strict initial object.* $\square$

Proposition 3.7. *If $\mathcal{B}$ has $\boldsymbol{T}$-stable finite limits, a $\boldsymbol{T}$-stable initial object, $\boldsymbol{T}$-stable coproducts and for each X in $\boldsymbol{T}$ the coproducts are disjoint and universal, then $\boldsymbol{T}\text{-}ind(\mathcal{A},\mathcal{B})$ has coproducts and they are disjoint and universal.*

Proof. By remark 3.1, $\boldsymbol{T}$-*ind*$(\mathcal{A},\mathcal{B})$ has coproducts and they are calculated pointwise at each X in $\boldsymbol{T}$. Since finite limits are pointwise too at every X and so is the initial object the result follows. $\square$

Proposition 3.8. *If $\mathcal{B}$ has $\boldsymbol{T}$-stable finite limits and $\boldsymbol{T}$-stable quotients of equivalence relations and for every X in $\boldsymbol{T}$ these quotients are universal then $\boldsymbol{T}$-*ind$(\mathcal{A},\mathcal{B})$ *has quotients of equivalence relations and they are universal.*

Proof. It is easy to see that an equivalence relation $F \underset{\tau}{\overset{\sigma}{\rightrightarrows}} G$ in $\boldsymbol{T}$-*ind*$(\mathcal{A},\mathcal{B})$ produces an equivalence relation $F^X \underset{\tau^X}{\overset{\sigma^X}{\rightrightarrows}} G^X$. Then proceed as before. $\square$

Proposition 3.9. *If $\mathcal{B}$ has $\boldsymbol{T}$-stable finite limits, $\boldsymbol{T}$-stable sups of subobjects and for every X in $\boldsymbol{T}$ they are universal then $\boldsymbol{T}$-*ind$(\mathcal{A},\mathcal{B})$ *has sups of subobjects and they are universal.* $\square$

Assume now that $\boldsymbol{T}$ has coproducts. Let $\mathcal{A}$ be a $\boldsymbol{T}$-indexed category and $\{X_\alpha\}_\alpha$ a family of objects in $\boldsymbol{T}$. Consider its coproduct $\langle X_\alpha \xrightarrow{i_\alpha} \coprod_\alpha X_\alpha \rangle_\alpha$. We obtain the functor $\langle i_\alpha^* \rangle : \mathcal{A}^{\coprod_\alpha X_\alpha} \to \prod_\alpha \mathcal{A}^{X_\alpha}$. We say that $\mathcal{A}$ distributes coproducts if for every family $\{X_\alpha\}_\alpha$ of objects in $\boldsymbol{T}$ the functor $\langle i_\alpha^* \rangle : \mathcal{A}^{\coprod_\alpha X_\alpha} \to \prod_\alpha \mathcal{A}^{X_\alpha}$ is an equivalence of categories with pseudo-inverse $\langle i_\alpha^* \rangle^-$. Notice that if we have a $\boldsymbol{T}$-indexed functor $F : \mathcal{A} \to \mathcal{B}$ and an arrow $f : Y \to X$ then the isomorphisms $F^{X_\alpha} i_\alpha^* \xrightarrow{\simeq} i_\alpha^* F^{\coprod_\alpha X_\alpha}$ induces an isomorphism

$$\begin{tikzcd}
\mathcal{A}^{\coprod_\alpha X_\alpha} \arrow[r, "\langle i_\alpha^* \rangle"] \arrow[d, "F^{\coprod_\alpha X_\alpha}"'] & \prod_\alpha \mathcal{A}^{X_\alpha} \arrow[d, "\prod_\alpha F^{X_\alpha}"] \arrow[dl, Rightarrow, "\simeq"'] \\
\mathcal{B}^{\coprod_\alpha X_\alpha} \arrow[r, "\langle i_\alpha^* \rangle"'] & \prod_\alpha \mathcal{B}^{X_\alpha}
\end{tikzcd}$$

and if both $\mathcal{A}$ and $\mathcal{B}$ distribute coproducts we obtain then a natural isomorphism

$$\begin{tikzcd}
\mathcal{A}^{\coprod_\alpha X_\alpha} \arrow[d, "F^{\coprod_\alpha X_\alpha}"'] \arrow[dr, Rightarrow, "\simeq"] & \prod_\alpha \mathcal{A}^{X_\alpha} \arrow[l, "\langle i_\alpha^* \rangle^-"'] \arrow[d, "\prod_\alpha F^{X_\alpha}"] \\
\mathcal{B}^{\coprod_\alpha X_\alpha} & \prod_\alpha \mathcal{B}^{X_\alpha} \arrow[l, "\langle i_\alpha^* \rangle^-"]
\end{tikzcd}$$

Definition 3.11. Let $\boldsymbol{T}$-IND be the full, 2 full subcategory of $\boldsymbol{T}$-*ind* whose objects are $\boldsymbol{T}$-indexed categories that distribute coproducts.

Remark 3.2. Since for any $\mathcal{A}$ and $\mathcal{B}$ in $\boldsymbol{T}$-IND we have $\boldsymbol{T}\text{-IND}(\mathcal{A},\mathcal{B}) = \boldsymbol{T}\text{-}ind(\mathcal{A},\mathcal{B})$ it is clear that the propositions above remain true when we are dealing with $\boldsymbol{T}$-IND.

The category $\mathcal{SET}$ is clearly an object of ***Top***-IND.

3.3 Continuous Families of Models

Let $\boldsymbol{P}$ be a pretopos, we define the ***Top***-indexed category $\mathcal{MOD}(\boldsymbol{P})$ of models over $\boldsymbol{P}$ as follows: Given a topological space X, let $\mathcal{MOD}(\boldsymbol{P})^X = \boldsymbol{Mod}_{Sh(X)}(\boldsymbol{P})$ and if $f : Y \to X$ define $f^* : \boldsymbol{Mod}_{Sh(X)}(\boldsymbol{P}) \to \boldsymbol{Mod}_{Sh(Y)}(\boldsymbol{P})$ as composition with $f^* : Sh(X) \to Sh(Y)$

$$(\boldsymbol{P} \xrightarrow{M} Sh(X)) \mapsto (\boldsymbol{P} \xrightarrow{M} Sh(X) \xrightarrow{f^*} Sh(Y)).$$

Since $f^* : Sh(X) \longrightarrow Sh(Y)$ has a right adjoint and it is left exact it is elementary, we have then that the composition with M is indeed a model. It is not hard to see that $\mathcal{MOD}(\boldsymbol{P})$ is in ***Top***-IND.

The ***Top***-indexed category $\mathcal{SET}$ is equivalent to $\mathcal{MOD}(\boldsymbol{P})$ for $\boldsymbol{P} = (\boldsymbol{Set}^{\boldsymbol{Set}_0})_{coh}$. Indeed, we know from Theorem 1.3 that we have an equivalence

$$\boldsymbol{Topos/Set}(Sh(X), Sh(\boldsymbol{P}, J)) \simeq \mathcal{MOD}(\boldsymbol{P})^X$$

where J is the precanonical topology on $\boldsymbol{P}$, and (see [8] 6.33)

$$\boldsymbol{Topos/Set}(Sh(X), \boldsymbol{Set}^{\boldsymbol{Set}_0}) \simeq Sh(X).$$

We have (see [11] 1.8)

Proposition 3.10. *The **Top**-indexed category $\mathcal{SET}$ has **Top**-stable finite limits, **Top**-stable colimits, **Top**-stable quotients of equivalence relations and they are universal at every X in **Top**, **Top**-stable sups of subobjects and they are universal at every X in **Top**.* $\square$

Corollary 3.11. *For every* $\boldsymbol{Top}$*-indexed category* $\mathcal{A}$ *the category* $\boldsymbol{Top}\text{-ind}(\mathcal{A},\mathcal{SET})$ *is an* ∞*-pretopos (in the sense of* [18]*, that is, it is left exact, has universal sups of small sets of subobjects, universal images, universal quotients of equivalence relations and universal disjoint coproducts).*

Proof. The result follows from Propositions 3.5, 3.7, 3.8 and Lemma 3.6. $\square$

It is shown in [11] that the $\boldsymbol{Top}$-indexed category $\mathcal{SET}$ is well powered, cowell powered and has small homs. We have

Proposition 3.12. *The* $\boldsymbol{Top}$*-indexed category* $\mathcal{MOD}(\boldsymbol{P})$ *has small homs at* 1.

Proof. Let $M \in \boldsymbol{Mod}(\boldsymbol{P})$, and $N \in \boldsymbol{Mod}_{Sh(X)}(\boldsymbol{P})$. Consider the diagram $\Gamma : El(M) \longrightarrow \boldsymbol{Top}/X$ such that $\Gamma(a \in MP) = NP$ where we consider NP as a local homeomorphism over X, and $\Gamma((a \in MP) \xrightarrow{\gamma} (b \in MP')) = (NP \xrightarrow{N\gamma} NP')$. Consider $\varprojlim_{El(M)} \Gamma(a \in MP) = \varprojlim_{El(M)} NP$ in $\boldsymbol{Top}/X$. Then for every $f : X \longrightarrow Y$ we have

$$\begin{array}{cl} h : f \longrightarrow \varprojlim_{El(M)} NP & \text{in } \boldsymbol{Top}/X \\ \hline \Big\langle \big\langle h_{(a\in MP)} : f \longrightarrow NP \big\rangle_{(a\in MP)} \Big\rangle_P & \text{in } \boldsymbol{Top}/X \end{array}$$

where for every $p : P \longrightarrow P'$ and any $a \in MP$ the diagram

$$\begin{tikzcd} f \arrow[rr, "h_{(a\in MP)}"] \arrow[dr, "h_{(Mp(a)\in MP')}"'] & & NP \arrow[dl, "Np"] \\ & NP' & \end{tikzcd}$$

commutes. Now,

$$\begin{array}{cl} \Big\langle \big\langle h_{(a\in MP)} : f \longrightarrow NP \big\rangle_{(a\in MP)} \Big\rangle_P & \text{in } \boldsymbol{Top}/X \\ \hline \Big\langle \big\langle k_{(a\in MP)} : 1 \longrightarrow f^*NP \big\rangle_{(a\in MP)} \Big\rangle_P & \text{in } \boldsymbol{Top}/Y \end{array}$$

where for every $p : P \longrightarrow P'$ and any $a \in MP$ the diagram

$$\begin{tikzcd} 1 \arrow[rr, "k_{(a\in MP)}"] \arrow[dr, "k_{(Mp(a)\in MP')}"'] & & f^*NP \arrow[dl, "Np"] \\ & f^*NP' & \end{tikzcd}$$

commutes. Then

$$\frac{\dfrac{\left\langle \left\langle k_{(a \in MP)} : 1 \longrightarrow f^*NP \right\rangle_{(a \in MP)} \right\rangle_P \quad \text{in } \boldsymbol{Top}/Y}{Y^*M \longrightarrow f^*N \quad \text{in } \mathcal{MOD}(\boldsymbol{P})^Y}}{f^*X^*M \longrightarrow f^*N \quad \text{in } \mathcal{MOD}(\boldsymbol{P})^Y}$$

In particular, for M, N in $\boldsymbol{Mod}(\boldsymbol{P})$ define $hom^1(M, N) = \varprojlim_{El(M)} NP$ in $\boldsymbol{Top}$. □

Notice that this gives a topology to the sets $\boldsymbol{Mod}(\boldsymbol{P})(M, N)$ for M, N in $\boldsymbol{Mod}(\boldsymbol{P})$. Indeed, for the topological space 1 we have the corresponding isomorphism

$$\boldsymbol{Top}(1, hom^1(M, N)) \longrightarrow \boldsymbol{Mod}(\boldsymbol{P})(M, N).$$

Notice that $hom^1(M, N)$ is a subspace of $\prod_{(a \in MP,P)} NP$. It is not hard to see that the topology for $\boldsymbol{Mod}(\boldsymbol{P})(M, N)$ has as subbasis sets of the form $U_{P,a,b}\{h : M \longrightarrow N \mid hP(a) = b\}$ with P in $\boldsymbol{P}$, $a \in MP$ and $b \in NP$.

Further analysis of smallness conditions for $\boldsymbol{Top}$-indexed categories of models will be done elsewhere.

3.4 Łos Categories

So far we have not dealt with arrows of the form f_* that allowed us to obtain the ultraproduct functors at the beginning of this chapter. We now take care of this.

Definition 3.12. Let $f : Y \to X$ be a morphism in $\boldsymbol{Top}$. We say that f is ultrafinite if $f_* : Sh(Y) \to Sh(X)$ preserves finite coproducts and epimorphisms.

Notice that $f : Y \to X$ ultrafinite means in particular that f_* is an elementary functor. Therefore, for every pretopos $\boldsymbol{P}$, composition with $f_* : Sh(Y) \to Sh(X)$ induces a functor $\mathcal{MOD}(\boldsymbol{P})^Y \to \mathcal{MOD}(\boldsymbol{P})^X$, also denoted by f_*, that is right adjoint to $f^* : \mathcal{MOD}(\boldsymbol{P})^X \to \mathcal{MOD}(\boldsymbol{P})^Y$.

As we mentioned before, given a discrete topological space I the usual embbeding $I \to \beta I$ into its Stone-Čech compactification is ultrafinite. We show this fact and give some more examples of ultrafinite functions below (see 3.5).

Definition 3.13. Given $\mathcal{A}$ in ***Top***-IND we say that $\mathcal{A}$ is a Los category if for every ultrafinite morphism $f : Y \to X$ the functor $f^* : \mathcal{A}^X \to \mathcal{A}^Y$ has a right adjoint $f_* : \mathcal{A}^Y \to \mathcal{A}^X$.

Given $\mathcal{A}$ and $\mathcal{B}$ in ***Top***-IND we say that a ***Top***-indexed functor $F : \mathcal{A} \to \mathcal{B}$ is a Los functor if for every ultrafinite $f : Y \to X$ in ***Top*** we have that the composition

$$F^X f_* \xrightarrow{\eta F^X f_*} f_* f^* F^X f_* \xrightarrow{\simeq} f_* F^Y f^* f_* \xrightarrow{f_* F^Y \epsilon'} f_* F^Y$$

is an isomorphism where η is the unit of $f^* \dashv f_* : \mathcal{B}^Y \to \mathcal{B}^X$, ϵ' is the counit of $f^* \dashv f_* : \mathcal{A}^Y \to \mathcal{A}^X$ and the middle isomorphism is induced by $f^* F^X \xrightarrow{\simeq} F^Y f_*$.

Given a pretopos $\boldsymbol{P}$ and an object P in $\boldsymbol{P}$ is is easy to see that the evaluation ***Top***-indexed functor $ev_P : \mathcal{MOD}(\boldsymbol{P}) \to \mathcal{SET}$ is a Los functor.

Definition 3.14. Let $\mathfrak{Los}$ be the 2-category whose objects are Los categories, its 1-cells Los functors and its 2-cells ***Top***-indexed natural transformations.

Thus $\mathfrak{Los}$ is a locally full subcategory of ***Top***-IND.

Proposition 3.13. *If $\mathcal{B}$ is a Los category that has*

*-**Top**-stable finite limits.*

*-**Top**-stable initial object strict at every X in **Top**.*

*-**Top**-stable finite coproducts that are disjoint and universal at every X.*

*-**Top**-stable quotients of equivalence relations universal at every X in **Top**.*

*Then for every Los category $\mathcal{A}$ the category $\mathfrak{Los}(\mathcal{A}, \mathcal{B})$ is a pretopos. Furthermore, the corresponding limits and colimits are calculated as in **Top**-IND$(\mathcal{A}, \mathcal{B})$.*

Proof. By Propositions 3.5, 3.7, 3.8 and Lemma 3.6 we have that ***Top***-IND$(\mathcal{A}, \mathcal{B})$ is a pretopos. All we have to show is that finite limits (coproducts, etc) of Los functors in ***Top***-IND$(\mathcal{A}, \mathcal{B})$ produce Los functors. Clearly the terminal functor $1 : \mathcal{A} \to \mathcal{B}$ is Los. Let F, G be functors in $\mathfrak{Los}(\mathcal{A}, \mathcal{B})$ and $f : Y \to X$ ultrafinite. Consider the following diagram

$$\begin{tikzcd}
(F \times G)^X f_* \arrow[r, "\simeq"] \arrow[d, "\eta(F \times G)^X f_*"'] & F^X f_* \times G^X f_* \arrow[d, "\eta F^X f_* \times \eta G^X f_*"] \\
f_* f^* (F \times G)^X f_* \arrow[r, "\simeq"'] \arrow[d, "\simeq"'] & f_* f^* F^X f_* \times f_* f^* G^X f_* \arrow[d, "\simeq"] \\
f_*(F \times G)^Y f^* f_* \arrow[r, "\simeq"'] \arrow[d, "f_*(F \times G)^Y \epsilon'"'] & f_* F^Y f^* f_* \times f_* G^Y f^* f_* \arrow[d, "f_* F^Y \epsilon' \times f_* G^Y \epsilon'"] \\
f_*(F \times G)^Y \arrow[r, "\simeq"'] & f_* F^Y \times f_* G^Y
\end{tikzcd}$$

where the top square commutes because $f_* f^*$ preserves finite products and η is natural, the one in the middle commutes by coherence and the bottom one commutes because $(F \times G)^Y$ is pointwise. Since F and G are Los the vertical composition on the right is an isomorphism. Therefore the vertical composition on the left is an isomorphism. A very similar argument shows that the pullback of Los functors is also Los. Therefore $\mathfrak{Los}(\mathcal{A}, \mathcal{B})$ has finite limits.

The initial functor $0 : \mathcal{A} \to \mathcal{B}$ is clearly Los. Showing that $\mathfrak{Los}(\mathcal{A}, \mathcal{B})$ has finite sums is a similar argument as before using the fact that f_* preserves finite sums. Finally we show that $\mathfrak{Los}(\mathcal{A}, \mathcal{B})$ has quotients of equivalence relations. Suppose that $F \underset{\tau}{\overset{\sigma}{\rightrightarrows}} G$ is an equivalence relation in $\mathfrak{Los}(\mathcal{A}, \mathcal{B})$. It is easy to see that (σ, τ) is then an equivalence relation in ***Top*-IND**. Consider $G \xrightarrow{\nu} H$ its quotient. We have to show that H is Los. Consider the following diagram

$$\begin{tikzcd}[column sep=large]
F^X f_* \arrow[r, shift left, "\sigma^X f_*"] \arrow[r, shift right, "\tau^X f_*"'] \arrow[d, "\eta F^X f_*"'] & G^X f_* \arrow[r, "\nu^X f_*"] \arrow[d, "\eta G^X f_*"] & H^X f_* \arrow[d, "\eta H^X f_*"] \\
f_* f^* F^X f_* \arrow[r, shift left, "f_* f^* \sigma^X f_*"] \arrow[r, shift right, "f_* f^* \tau^X f_*"'] \arrow[d, "\simeq"'] & f_* f^* G^X f_* \arrow[r, "f_* f^* \nu^X f_*"'] \arrow[d, "\simeq"] & f_* f^* H^X f_* \arrow[d, "\simeq"] \\
f_* F^Y f^* f_* \arrow[r, shift left, "f_* \sigma^Y f^* f_*"] \arrow[r, shift right, "f_* \tau^Y f^* f_*"'] \arrow[d, "f_* F^Y \epsilon'"'] & f_* G^Y f^* f_* \arrow[r, "f_* \nu^Y f^* f_*"'] \arrow[d, "f_* G^Y \epsilon'"] & f_* H^Y f^* f_* \arrow[d, "f_* H^Y \epsilon'"] \\
f_* F^Y \arrow[r, shift left, "f_* \sigma^Y"] \arrow[r, shift right, "f_* \tau^Y"'] & f_* G^Y \arrow[r, "f_* \nu^Y"'] & f_* H^Y .
\end{tikzcd}$$

It is not hard to prove that the diagram commutes. Since f_* preserves epimorphisms we have that $f_* \nu^Y$ is an epi. Since

$$\begin{tikzcd}
f_* F^Y \arrow[r, "f_* \sigma^Y"] \arrow[d, "f_* \tau^Y"'] & f_* G^Y \arrow[d, "f_* \nu^Y"] \\
f_* G^Y \arrow[r, "f_* \nu^Y"'] & f_* H^Y
\end{tikzcd}$$

is a pullback we have that the last row in the diagram is a coequalizer. Since the first row is also a coequalizer and the first two vertical compositions are isomorphisms we conclude that the third vertical composition is also an isomorphism. So we have that H is in $\mathfrak{Los}(\mathcal{A}, \mathcal{B})$. $\square$

It is easy to see that if $\mathcal{B}$ satisfies the conditions of Proposition 3.13 and $F : \mathcal{A} \to \mathcal{C}$ is a Los functor between Los categories then $\mathfrak{Los}(F, \mathcal{B}) : \mathfrak{Los}(\mathcal{C}, \mathcal{B}) \to \mathfrak{Los}(\mathcal{A}, \mathcal{B})$ is an elementary functor. We therefore obtain a functor $\mathfrak{Los}^{op} \to \boldsymbol{PRETOP}$.

3.5 Characterization of Ultrafinite Functions

We now turn our attention to ultrafinite functions in ***Top***.

In what follows we will use the well known equivalent descriptions of $\mathcal{SET}^X$ as the usual $Sh(X)$ and as the category LH/X of local homeomorphisms over X, for a topological space X. We use the usual equivalences $\Gamma : LH/X \longrightarrow \mathcal{SET}^X$ and $L : Sh(Y) \longrightarrow LH/Y$ (see [2] for example).

Lemma 3.14. *Let $f : X \to Y$ be a continuous function then $f_* : Sh(X) \to Sh(Y)$ preserves the initial object if and only if $f(X)$ is dense in Y.*

Proof. Suppose first that f_* preserves initial object. Let V be a nonempty open set of Y, and let $\mathbf{0}$ represent the initial sheaf, then $f_*(\mathbf{0})(V) = \emptyset$. That is, $\mathbf{0}(f^{-1}V) = \emptyset$. Therefore, $f^{-1}V$ can not be the empty set, and then $V \cap f(X) \neq \emptyset$

In the other direction, suppose f is dense. Let V be open in Y, since $f(X)$ is dense in Y, we have that $f^{-1}(V) \neq \emptyset$. Therefore $\mathbf{0}(f^{-1}(U)) = \emptyset$. So $f_*(\mathbf{0}) = \mathbf{0}$. $\square$

For the rest of the section rather that working with $f_* : Sh(X) \to Sh(Y)$, we will be working with $LH/X \xrightarrow{\Gamma} Sh(X) \xrightarrow{f_*} Sh(Y) \xrightarrow{L} LH/Y$. If we have

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

in LH/X, then we have that the map

$$\varinjlim_{V \ni y} \Gamma(E,p)(f^{-1}(V)) \xrightarrow{\quad Lf_*\Gamma(h) \quad} \coprod_{y \in Y} \varinjlim_{V \ni y} \Gamma(E',p')(f^{-1}(V))$$

is such that $[s \in \Gamma(E,p)(f^{-1}(V))]_y \mapsto [h \circ s \in \Gamma(E',p')(f^{-1}(V))]_y$

Lemma 3.15. *Let $f : X \to Y$ be a continuous function with dense image. Then $f_* : Sh(X) \to Sh(Y)$ preserves finite coproducts if and only if for every open $V \subset Y$ and every $y \in V$, whenever $f^{-1}(V)$ is the union of two disjoint open sets of X, there exists $W \subset Y$ open with $y \in W$ such that $f^{-1}(W)$ is contained in one of them.*

Proof. Suppose f_* preserves finite coproducts, therefore $Lf_*\Gamma$ preserves finite coproducts. Consider the following coproduct in LH/X

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "id_X"'] & X \coprod X \arrow[d, "\langle id_X, id_X \rangle"] & X \arrow[l, "i_2"'] \arrow[dl, "id_X"] \\ & X & \end{tikzcd}$$

Since $Lf_*\Gamma$ preserves finite coproducts, we have that the induced continuous function

$$Lf_*\Gamma(X, id_X)\coprod Lf_*\Gamma(X, id_X) \xrightarrow{\cong} Lf_*\Gamma(X\coprod X, \langle id_X, id_X\rangle)$$

is a homeomorphism. Take $V \subset Y$ an open set, $y \in V$, and suppose that $f^{-1}(V) = A \cup B$ with A and B open and disjoint. Define $s : f^{-1}(V) \longrightarrow X \coprod X$ such that $s|_A$ is the inclusion of A into the first factor, and $s|_B$ is the inclusion of B into the second factor. Then s is continuous and $[s]_y \in Lf_*\Gamma(X \coprod X, \langle id_X, id_X\rangle)$. Therefore there exists an open set W of Y, and a continuous function $t : f^{-1}(W) \longrightarrow X$ such that one of the following diagrams commute

$$\begin{tikzcd} f^{-1}(W) \arrow[r, "t"] \arrow[d] & X \arrow[d, "i_1"] \\ f^{-1}(V) \arrow[r, "s"'] & X \coprod X \end{tikzcd} \qquad \begin{tikzcd} f^{-1}(W) \arrow[r, "t"] \arrow[d] & X \arrow[d, "i_2"] \\ f^{-1}(V) \arrow[r, "s"'] & X \coprod X \end{tikzcd}$$

In any case, we have $f^{-1}(W) \subset A$ or $f^{-1}(W) \subset B$.

In the other direction, consider the coproduct

$$\begin{tikzcd} E \arrow[r, "i_E"] \arrow[dr, "p"'] & E \coprod E' \arrow[d, "\langle p, p' \rangle"] & E' \arrow[l, "i_{E'}"'] \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

in the category LH/X. Then we induce the unique morphism φ that makes the diagram

$$\begin{tikzcd}
Lf_*\Gamma(E,p) \arrow[r] \arrow[dr] & Lf_*\Gamma(E,p)\coprod Lf_*\Gamma(E',p') \arrow[d, "\varphi"] & Lf_*\Gamma(E',p') \arrow[l] \arrow[dl] \\
 & Lf_*\Gamma(E\coprod E',\langle p,p'\rangle) &
\end{tikzcd}$$

commute. We have to show that φ is a homeomorphism. First we show that φ is monomorphic. Suppose $\varphi([f^{-1}(V) \xrightarrow{s} E]_y) = \varphi([f^{-1}(W) \xrightarrow{s} E']_z)$. Then it is clear that $y = z$ and that

$$[f^{-1}(V) \xrightarrow{s} E \xrightarrow{i_E} E\coprod E']_y = [f^{-1}(W) \xrightarrow{t} E' \xrightarrow{i_{E'}} E\coprod E']_y$$

Therefore, there exists $U \subset Y$ open such that $y \in U \subset V \cap W$, and $r : f^{-1}(U) \longrightarrow E\coprod E'$ such that

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, "s"'] & f^{-1}(U) \arrow[l] \arrow[r] \arrow[d, "r"] & f^{-1}(W) \arrow[d, "t"] \\
E \arrow[r, "i_E"'] & E\coprod E' & E' \arrow[l, "i'_E"]
\end{tikzcd}$$

commutes. Suppose $x \in f^{-1}(U)$. Then $r(x) \in E$ and $r(x) \in E'$, a contradiction. Therefore $f^{-1}(U) = \emptyset$. But U is open and nonempty, and $f(X)$ is dense in Y, therefore $f^{-1}(U)$ is nonempty, another contradiction. Therefore we conclude that it is not possible that $\varphi([f^{-1}(V) \xrightarrow{s} E]_y) = \varphi([f^{-1}(W) \xrightarrow{s} E']_z)$.

Suppose now that $\varphi([f^{-1}(V) \xrightarrow{s} E]_y) = \varphi([f^{-1}(W) \xrightarrow{s} E]_z)$. Then we proceed as before, so $y = z$ and we can find U open in Y with $y \in U$ and $U \subset V \cap W$ and $r : f^{-1}(U) \longrightarrow E\coprod E'$ such that

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, "s"'] & f^{-1}(U) \arrow[l] \arrow[r] \arrow[d, "r"] & f^{-1}(W) \arrow[d, "t"] \\
E \arrow[r, "i_E"'] & E\coprod E' & E \arrow[l, "i_E"]
\end{tikzcd}$$

commutes. But this means that $Im(r) \subset E$, and

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d] & f^{-1}(V) \arrow[d, "s"] \\ f^{-1}(W) \arrow[r, "t"'] & E \end{tikzcd}$$

therefore $[f^{-1}(V) \xrightarrow{s} E]_y = [f^{-1}(W) \xrightarrow{s} E]_y$, and φ is mono.

Now, take $[f^{-1}(V) \xrightarrow{s} E \coprod E']_y \in Lf_*\Gamma(E \coprod E', \langle p, p' \rangle)$, then $f^{-1}(V) = s^{-1}(E) \cup s^{-1}(E')$ with $s^{-1}(E)$ and $s^{-1}(E')$ open and disjoint. Therefore there is a $W \subset Y$ open such that $y \in W$, and $f^{-1}(W) \subset s^{-1}(E)$ or $f^{-1}(W) \subset s^{-1}(E')$. If $f^{-1}(W) \subset s^{-1}(E)$. Then $\varphi([f^{-1}(W) \xrightarrow{s|_{f^{-1}(W)}} E]_y) = [s]_y$. The other case is similar. Finally, φ is open because it is a local homeomorphism. $\square$

If we consider

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

in LH/X as before, then $Lf_*\Gamma(h)$ is an epimorphism iff for every $y \in Y$, every V open in Y with $y \in V$ and any $s : f^{-1}(V) \longrightarrow E'$ such that $p' \circ s$ equals the inclusion of $f^{-1}(V)$ in X, then there exist W open in Y with $y \in W$ and $t : f^{-1}(W) \longrightarrow E$ such that

$$\begin{tikzcd} f^{-1}(W) \arrow[r, "t"] \arrow[d] & E \arrow[d, "h"] \\ f^{-1}(V) \arrow[r, "s"'] & E' \end{tikzcd}$$

commutes, where the left vertical arrow is the inclusion.

Lemma 3.16. *If $f : X \to Y$ is a continuous function, then $f_* : Sh(X) \to Sh(Y)$ preserves epimorphisms if and only if for every $V \subset Y$ open, $y \in V$ and every open cover $\{U_\alpha\}_{\alpha \in A}$ of $f^{-1}(V)$, there exist an open W of Y with $y \in W \subset V$, and a disjoint open cover $\{W_\alpha\}_{\alpha \in A}$ of $f^{-1}(W)$ such that for every α we have that $W_\alpha \subset U_\alpha$*

Proof. Consider a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

with p and p' local homeomorphisms and h onto. Take V open in Y and $y \in V$. Suppose that $s : f^{-1}(V) \longrightarrow E'$ is such that $p' \circ s$ equals the inclusion $i : f^{-1}(V) \longrightarrow X$. Since s is a local homeomorphism, it is open. Therefore $s(f^{-1}(V))$ is open in E', and $s : f^{-1}(V) \longrightarrow s(f^{-1}(V))$ is a homeomorphism with inverse p'. Since h is continuous we have that $h^{-1}(s(f^{-1}(V)))$ is open in E. So we have the following commutative diagram

$$\begin{tikzcd} h^{-1}(s(f^{-1}(V))) \arrow[r, "h"] \arrow[dr, "p"'] & s(f^{-1}(V)) \arrow[r, "p'"] \arrow[d, "p'"] & f^{-1}(V) \arrow[dl, "i"] \\ & X & \end{tikzcd}$$

where the composition at the top is clearly onto. It is clear that it is enough to find $W \subset V$ with $y \in W$ and $t : f^{-1}(W) \longrightarrow h^{-1}(s(f^{-1}(V)))$ such that

$$\begin{tikzcd} f^{-1}(W) \arrow[rr, "t"] \arrow[dr] & & h^{-1}(s(f^{-1}(V))) \arrow[dl, "p' \circ h"] \\ & f^{-1}(V) & \end{tikzcd}$$

commutes. So, we may suppose that we have a local homeomorphism $q : E'' \longrightarrow f^{-1}(V)$ that is onto and we want to find $W \subset Y$ with $y \in W$ and $t : f^{-1}(W) \longrightarrow E''$ such that

$$\begin{tikzcd} f^{-1}(W) \arrow[rr, "t"] \arrow[dr] & & E'' \arrow[dl, "q"] \\ & f^{-1}(V) & \end{tikzcd}$$

commutes.

For every $x \in f^{-1}(V)$ choose $U_x \subset f^{-1}(V)$ open, $U'_x \subset E''$ open such that $x \in U_x$ and $q : U'_x \longrightarrow U_x$ is a homeomorphism. Then, $\{U_x\}_{x \in f^{-1}(V)}$ is an open cover of $f^{-1}(V)$. Therefore there exist $W \subset V$ open with $y \in W$ and a disjoint open cover $\{W_x\}_{x \in f^{-1}(V)}$ of $f^{-1}(W)$ such that $W_x \subset U_x$ for every $x \in f^{-1}(V)$. Define $t_x = (q|_{U'_x})^{-1}|_{W_x} : W_x \longrightarrow E''$. Since $\{W_x\}_{x \in f^{-1}(V)}$ are disjoint and clopen in $f^{-1}(W)$ it is clear that we can put them together to obtain the continuous function $t : f^{-1}(W) \longrightarrow E''$ such that $t|_{W_x} = t_x$. t has the required property. $\square$

We put Lemmas 3.14, 3.15 and and 3.16 together in the following proposition.

Proposition 3.17. *A continuous function $f : X \to Y$ is ultrafinite if and only if f satisfies the following conditions:*

(1) *$f(X)$ is dense in Y.*

(2) *For every open V of Y and any $y \in V$, if $f^{-1}(V) = A \cup B$ with A and B open and disjoint, then there exists an open $W \subset V$ with $y \in W$ such that $f^{-1}(W) \subset A$ or $f^{-1}(W) \subset B$.*

(3) *For every open V of Y, any $y \in V$ and any open cover $\{U_\alpha\}_{\alpha \in A}$ of $f^{-1}(V)$, there exists an open $W \subset V$ with $y \in W$ and a disjoint open cover $\{W_\alpha\}_{\alpha \in A}$ of $f^{-1}(W)$ such that for every $\alpha \in A$ we have that $W_\alpha \subset U_\alpha$.*

$\square$

Proposition 3.18. *Given a discrete topological space I, the usual embedding $\xi I : I \to \beta I$ into its Stone-Čech compactification is ultrafinite.*

Proof. Since ξI is dense we have by Lemma 3.14 that $\xi I_* : Sh(I) \to Sh(\beta I)$ preserves the initial object. Take a basic open J^* and an element $\mathcal{U} \in J^*$ and assume that $\xi I^{-1}(J^*) = J_1 \cup J_2$ with $J_1 \cap J_2 = \emptyset$. Since $\xi I^{-1}(J^*) = J$ we have $J_1 \cup J_2 \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter that means that $J_1 \in \mathcal{U}$ or $J_2 \in \mathcal{U}$. That is $\mathcal{U} \in J_1^*$ or $\mathcal{U} \in J_2^*$ and $\xi I^{-1}(J_1^*) \subset J_k$ for $k = 1$ or for $k = 2$. By Lemma 3.15 we have that ξI_* preserves finite coproducts. Using Zorn's lemma it can be shown that for any family $\{I_\alpha\}$ of subsets of I we can find a disjoint family $\{J_\alpha\}$ such that $\bigcup_\alpha J_\alpha = \bigcup_\alpha I_\alpha$ and for every α, $J_\alpha \subset I_\alpha$. So given a basic open J^*, a point $\mathcal{U} \in J^*$ and an open covering $\{I_\alpha\}$ of ξI^{-1} we simply replace the family $\{I_\alpha\}$ with a disjoint family $\{J_\alpha\}$ with the same union such that $J_\alpha \subset I_\alpha$ for all α. By Lemma 3.16 we have that ξI_* preserves epis. □

We need not take all of βI. If we take a non principal ultrafilter $\mathcal{U}$ on I and consider the topological space $\xi I(I) \cup \{\mathcal{U}\}$ with the topology it inherits from βI we have that the resulting embbeding $I \to \xi I(I) \cup \{\mathcal{U}\}$ is ultrafinite. We normally identify $\xi I(I)$ with I, denote the element corresponding to $\mathcal{U}$ by $a_{\mathcal{U}}$ and denote the resulting space by $I_{\mathcal{U}}$.

Another example of an ultrafinite function is the following. Let $\boldsymbol{D}$ be a directed category. Consider the topological space $X_{\boldsymbol{D}}$ whose elements are the objects of $\boldsymbol{D}$ and give $X_{\boldsymbol{D}}$ the Alexandroff topology, that is the sets of the form $\uparrow(d) = \{d' |$ there exists an arrow $d \to d'\}$ form a basis for the topology. Consider the topological space $X_{\boldsymbol{D}} \cup \{p\}$ where $p \notin X_{\boldsymbol{D}}$ and with basis $\{\uparrow(d) \cup \{p\}\}_{d \in \boldsymbol{D}}$. Notice that we need $\boldsymbol{D}$ directed for the given family to form a basis. We have an obvious continuous function $X_{\boldsymbol{D}} \to X_{\boldsymbol{D}} \cup \{p\}$. It is not hard to see that this function is ultrafinite.

Chapter 4

Algebras

4.1 2-Monads

We will consider several monads. In this section we give the definitions we will be using later to fix the notation. We follow the notation of [5].

Given a 2-category $\boldsymbol{A}$, a strict 2-monad on $\boldsymbol{A}$ is a 2-endofunctor $T : \boldsymbol{A} \to \boldsymbol{A}$ together with 2-natural transformations $\eta : 1 \to T$ and $\mu : TT \to T$ such that the usual diagrams

$$\begin{tikzcd} TA \arrow[r, "T\eta A"] \arrow[dr, "1_{TA}"'] & TTA \arrow[d, "\mu A"] & TA \arrow[l, "\eta TA"'] \arrow[dl, "1_{TA}"] \\ & TA & \end{tikzcd} \qquad \begin{tikzcd} TTTA \arrow[r, "T\mu A"] \arrow[d, "\mu TA"'] & TA \arrow[d, "\mu A"] \\ TTA \arrow[r, "\mu A"'] & TA \end{tikzcd}$$

commute on the nose. Given a strict 2-monad $\boldsymbol{T} = (T, \eta, \mu)$ a strict algebra is a pair (A, Φ) where A is an object of $\boldsymbol{A}$ and $\Phi : TA \to A$ is a 1-cell of $\boldsymbol{A}$ such that the ususal diagrams

$$\begin{tikzcd} A \arrow[r, "\eta A"] \arrow[dr, "1_A"'] & TA \arrow[d, "\Phi"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} TTA \arrow[r, "\mu A"] \arrow[d, "T\Phi"'] & TA \arrow[d, "\Phi"] \\ TA \arrow[r, "\Phi"'] & A \end{tikzcd}$$

commute on the nose. Given algebras (A, Φ) and (B, Ψ) a morphism of algebras is a pair $(H, \varphi) : (A, \Phi) \to (B, \Psi)$ where $H : A \to B$ is a 1-cell in $\boldsymbol{A}$ and φ is an invertible two cell

$$\begin{tikzcd}
T\boldsymbol{A} \arrow[r, "\Phi"] \arrow[d, "TH"'] \arrow[dr, phantom, "\swarrow\varphi"] & \boldsymbol{A} \arrow[d, "H"] \\
T\boldsymbol{B} \arrow[r, "\Psi"'] & \boldsymbol{B}
\end{tikzcd}$$

satisfying the coherence axioms

$$\begin{tikzcd}
TT\boldsymbol{A} \arrow[r, "TTH"] \arrow[d, "\mu\boldsymbol{A}"'] & TT\boldsymbol{B} \arrow[d, "\mu\boldsymbol{B}"] \\
T\boldsymbol{A} \arrow[r, "TH"'] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\varphi\nearrow"] & T\boldsymbol{B} \arrow[d, "\Psi"] \\
\boldsymbol{A} \arrow[r, "H"'] & \boldsymbol{B}
\end{tikzcd}
=
\begin{tikzcd}
TT\boldsymbol{A} \arrow[r, "TTH"] \arrow[d, "T\Phi"'] \arrow[dr, phantom, "T\varphi\nearrow"] & TT\boldsymbol{B} \arrow[d, "T\Psi"] \\
T\boldsymbol{A} \arrow[r, "TH"'] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\varphi\nearrow"] & T\boldsymbol{B} \arrow[d, "\Psi"] \\
\boldsymbol{A} \arrow[r, "H"'] & \boldsymbol{B}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\boldsymbol{A} \arrow[r, "H"] \arrow[d, "\eta\boldsymbol{A}"'] & \boldsymbol{B} \arrow[d, "\eta\boldsymbol{B}"] \\
T\boldsymbol{A} \arrow[r, "TH"'] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\varphi\nearrow"] & T\boldsymbol{B} \arrow[d, "\Psi"] \\
\boldsymbol{A} \arrow[r, "H"'] & \boldsymbol{B}.
\end{tikzcd}
= id_H$$

When φ is an identity we say that the morphism (H, φ) is strict.

We consider the 2-category $\boldsymbol{T}$-ALG whose objects are strict algebras (A, Φ), whose 1-cells are morphisms of algebras $(H, \varphi) : (A, \Phi) \to (B, \Psi)$ and whose 2-cells τ :

$(H, \varphi) \to (K, \psi)$ are 2-cells $\tau : H \to K$ in $\boldsymbol{A}$ such that

$$
\begin{tikzcd}
TA \arrow[r, bend left, "TK"] \arrow[r, bend right, "TH"'] \arrow[r, phantom, "\Uparrow T\tau"] \arrow[d, "\Phi"'] & TB \arrow[d, "\Psi"] \\
A \arrow[r, "H"'] \arrow[ur, phantom, "\nearrow\varphi"] & B.
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
TA \arrow[r, "TK"] \arrow[d, "\Phi"'] & TB \arrow[d, "\Psi"] \\
A \arrow[r, bend left, "K"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Uparrow\tau"] \arrow[ur, phantom, "\nearrow\psi"] & B
\end{tikzcd}
$$

We have the 2-subcategory $\boldsymbol{T}\text{-}ALG_s$ of $\boldsymbol{T}\text{-}ALG$ where we restrict the morphisms to strict morphisms. Thus the inclusion 2-functor is not full but it is locally full and faithful.

4.2 Functorial Weak (Co)Limits

In this section we review some of the folklore of weak limits.

Let $\boldsymbol{A}$ be a category. For every object A in $\boldsymbol{A}$ we have the usual forgetful functor $U_A : A/\boldsymbol{A} \to \boldsymbol{A}$.

Definition 4.1. A functorial weak initial object in $\boldsymbol{A}$ is a pair (Z, F) with Z an object of $\boldsymbol{A}$ and $F : \boldsymbol{A} \to Z/\boldsymbol{A}$ a functor such that the diagram

$$
\begin{tikzcd}
\boldsymbol{A} \arrow[rr, "F"] \arrow[dr, "1_{\boldsymbol{A}}"'] & & Z/\boldsymbol{A} \arrow[dl, "U_Z"] \\
& \boldsymbol{A} &
\end{tikzcd}
$$

commutes. We say that $\boldsymbol{A}$ has a functorial weak initial object if such a pair (Z, F) exists.

Functorial weak terminal object is defined dually.

If (Z, F) is a functorial weak initial object in $\boldsymbol{A}$ then clearly Z is a weak initial object in $\boldsymbol{A}$. Furthermore, for every arrow $a : A \to A'$ the diagram

$$
\begin{tikzcd}
Z \arrow[rr, "FA"] \arrow[dr, "FA'"'] & & A \arrow[dl, "a"] \\
& A' &
\end{tikzcd}
$$

commutes. In particular, considering $FA : Z \to A$ as an arrow in $\boldsymbol{A}$ we have that

$$\begin{tikzcd} Z \arrow[rr, "FZ"] \arrow[dr, "FA"'] & & Z \arrow[dl, "FA"] \\ & A & \end{tikzcd} \tag{4.1}$$

commutes.

Lemma 4.1. *If (Z, F) is a functorial weak initial object in $\boldsymbol{A}$ then $FZ : Z \to Z$ is an idempotent.*

Proof. For $A = Z$ in 4.1 we obtain $FZ \circ FZ = FZ$. □

Proposition 4.2. *If $\boldsymbol{A}$ has a functorial weak initial object (Z, F) and split idempotents then $\boldsymbol{A}$ has an initial object.*

Proof. From Lemma 4.1 FZ is an idempotent. Consider a splitting

$$\begin{tikzcd} Z \arrow[rr, "FZ"] \arrow[dr, "e"'] & & Z \\ & S \arrow[ur, "m"'] & \end{tikzcd}$$

Since

$$\begin{tikzcd} Z \arrow[rr, "FZ"] \arrow[dr, "FS"'] & & Z \\ & S \arrow[ur, "m"'] & \end{tikzcd}$$

commutes, we have $m \circ FS = FZ = m \circ e$. Since m is mono, $FS = e$. Given A in $\boldsymbol{A}$ we have the arrow $S \xrightarrow{m} Z \xrightarrow{FA} A$. Suppose now that we have another arrow $g : S \to A$. Consider the diagram

$$\begin{tikzcd} & & Z \arrow[dd, "FA"] \\ Z \arrow[urr, "FZ"] \arrow[r, shift left, "e"] \arrow[r, shift right, "FS"'] \arrow[drr, "FA"'] & S \arrow[ur, "m"] \arrow[dr, "g"'] & \\ & & A \end{tikzcd}$$

Both triangles on the left commute and the exterior triangle also commutes, therefore $FA \circ m \circ e = g \circ e$. Since e is epi we have $FA \circ m = g$. This shows S is initial. □

Let $\Gamma : \boldsymbol{I} \to \boldsymbol{A}$ be a diagram. Define the category $\boldsymbol{Cocone}(\Gamma)$ of cocones over Γ. That is, the objects of $\boldsymbol{Cocone}(\Gamma)$ are cocones $\langle \Gamma I \xrightarrow{f_I} A \rangle_I$ and a morphism

$$a : \langle \Gamma I \xrightarrow{f_I} A \rangle_I \to \langle \Gamma I \xrightarrow{f'_I} A' \rangle_I$$

is an arrow $a : A \to A'$ such that for every I in $\boldsymbol{I}$ the diagram

$$\begin{tikzcd} A \arrow[rr, "a"] & & A' \\ & \Gamma I \arrow[ul, "f_I"] \arrow[ur, "f'_I"'] & \end{tikzcd}$$

commutes. There is an obvious forgetful functor $\boldsymbol{Cocone}(\Gamma) \to \boldsymbol{A}$ and a weak colimit cocone for Γ in $\boldsymbol{A}$ is clearly a weak initial object in the category $\boldsymbol{Cocone}(\Gamma)$ and vice versa.

Definition 4.2. A functorial weak colimit for Γ in $\boldsymbol{A}$ is a functorial weak initial object in the category $\boldsymbol{Cocone}(\Gamma)$.

Functorial weak limits are defined dually.

A functorial weak colimit for Γ in $\boldsymbol{A}$ clearly gives a weak colimit cocone for Γ.

Lemma 4.3. *If the category $\boldsymbol{A}$ has split idempotents then the category $\boldsymbol{Cocone}(\Gamma)$ has split idempotents.* □

Proposition 4.4. *If a category $\boldsymbol{A}$ has split idempotents and a functorial weak colimit for a diagram $\Gamma : \boldsymbol{I} \to \boldsymbol{A}$ then Γ has a colimit in $\boldsymbol{A}$.*

Proof. By Lemma 4.3, $\boldsymbol{Cocone}(\Gamma)$ has split idempotents and we are supposing that $\boldsymbol{Cocone}(\Gamma)$ has a functorial weak initial object. Then by Proposition 4.2, $\boldsymbol{Cocone}(\Gamma)$ has an initial object. This initial object is a colimit cocone for Γ in $\boldsymbol{A}$. □

4.3 Pseudo-retractions

Suppose now we have functors $\boldsymbol{A} \xrightarrow{H} \boldsymbol{B}$ and $\boldsymbol{B} \xrightarrow{R} \boldsymbol{A}$ and a natural transformation

$$\begin{tikzcd}
\boldsymbol{A} \arrow[rr, "H"] \arrow[dr, "1_{\boldsymbol{A}}"'] & \arrow[d, phantom, "\xleftarrow{\theta}"] & \boldsymbol{B} \arrow[dl, "R"] \\
& \boldsymbol{A} &
\end{tikzcd} \tag{4.2}$$

Proposition 4.5. *In the above situation, if $\boldsymbol{B}$ has a functorial weak initial object then $\boldsymbol{A}$ has a functorial weak initial object.*

Proof. Assume (Z, F) is a functorial weak initial object for $\boldsymbol{B}$. Given $a : A \to A'$ in $\boldsymbol{A}$ we have the commutative diagram

$$\begin{tikzcd}
Z \arrow[rr, "FHA"] \arrow[dr, "FHA'"'] & & HA \arrow[dl, "Ha"] \\
& HA' &
\end{tikzcd}$$

in $\boldsymbol{B}$. Applying R and using the naturality of θ we obtain the commutative diagram

$$\begin{tikzcd}
RZ \arrow[rr, "\theta A \circ R(FHA)"] \arrow[dr, "\theta A' \circ R(FHA')"'] & & A \arrow[dl, "a"] \\
& A' &
\end{tikzcd}$$

Therefore $(RZ, \theta(\_) \circ R(FH(\_)))$ is a functorial weak initial object in $\boldsymbol{A}$. $\square$

Remark 4.1. Notice that for the dual, that is for functorial weak terminal object we need to reverse the natural transformation θ.

Assume now that θ in 4.2 is a natural isomorphism and let $\Gamma : \boldsymbol{I} \to \boldsymbol{A}$ be a diagram. We can induce then a functor $R' : \boldsymbol{Cocone}(H\Gamma) \to \boldsymbol{Cocone}(\Gamma)$ such that $R'\langle H\Gamma I \xrightarrow{g_I} B\rangle_I = \langle \Gamma I \xrightarrow{\theta\Gamma I^{-1}} RH\Gamma I \xrightarrow{Rf_I} RB\rangle_I$ and $R'b = Rb$ for every $b : \langle \Gamma I \xrightarrow{g_I} B\rangle_I \to \langle \Gamma I \xrightarrow{g'_I} B'\rangle_I$ in $\boldsymbol{Cocone}(H\Gamma)$. We have that H induces a functor $H' : \boldsymbol{Cocone}(\Gamma) \to \boldsymbol{Cocone}(H\Gamma)$ such that $H'\langle \Gamma I \xrightarrow{f_I} A\rangle_I = \langle H\Gamma I \xrightarrow{Hf_I} HA\rangle_I$

and $H'a = Ha$ for every $a : \langle \Gamma I \xrightarrow{f_I} A\rangle_I \to \langle \Gamma I \xrightarrow{f'_I} A'\rangle_I$ in $\boldsymbol{Cocone}(\Gamma)$. We can induce a natural isomorphism

$$\begin{tikzcd} \boldsymbol{Cocone}(\Gamma) \arrow[rr, "H'"] \arrow[dr, "1_{\boldsymbol{Cocone}(\Gamma)}"'] & \arrow[d, phantom, "\xleftarrow{\widehat{\theta}}"] & \boldsymbol{Cocone}(H\Gamma) \arrow[dl, "R'"] \\ & \boldsymbol{Cocone}(\Gamma) & \end{tikzcd} \tag{4.3}$$

such that $\widehat{\theta}\langle \Gamma I \xrightarrow{f_I} A\rangle_I = \theta A : \langle \Gamma I \xrightarrow{\theta\Gamma I^{-1}} RH\Gamma I \xrightarrow{RHf_I} RHA\rangle_I \to \langle \Gamma I \xrightarrow{f_I} A\rangle_I$

Theorem 4.6. *If θ in 4.2 is a natural isomorphism, then any diagram $\Gamma : \boldsymbol{I} \to \boldsymbol{A}$ such that $\boldsymbol{I} \xrightarrow{\Gamma} \boldsymbol{A} \xrightarrow{H} \boldsymbol{B}$ has a functorial weak colimit (functorial weak limit) in $\boldsymbol{B}$ has a functorial weak colimit (functorial weak limit) in $\boldsymbol{A}$. In particular, if $\boldsymbol{A}$ has split idempotents then Γ has a colimit (limit) in $\boldsymbol{A}$.*

Proof. Since $\boldsymbol{I} \xrightarrow{\Gamma} \boldsymbol{A} \xrightarrow{H} \boldsymbol{B}$ has a functorial weak colimit in $\boldsymbol{B}$ we have that the category $\boldsymbol{Cocone}(H\Gamma)$ has a functorial weak initial object. Since θ is a natural isomorphism we can induce $\widehat{\theta}$ in 4.3. By Proposition 4.5 we have that $\boldsymbol{Cocone}(\Gamma)$ has a functorial weak initial object, that is Γ has a functorial weak colimit in $\boldsymbol{A}$. If $\boldsymbol{A}$ has split idempotents then by Lemma 4.3, $\boldsymbol{Cocone}(\Gamma)$ has split idempotents. By Proposition 4.2 $\boldsymbol{Cocone}(\Gamma)$ has an initial object. This initial object is a colimit for Γ in $\boldsymbol{A}$. $\square$

Remark 4.2. In the cases we are going to consider the category $\boldsymbol{B}$ will have split idempotents. This implies that $\boldsymbol{A}$ has split idempotents (provided θ is a natural isomorphism). Indeed, if $a : A \to A$ is an idempotent in $\boldsymbol{A}$ then Ha is an idempotent in $\boldsymbol{B}$. Splitting Ha and applying R we obtain a splitting of RHa, use now that θ is iso. We will also have a colimit (limit) of the diagram $\boldsymbol{I} \xrightarrow{\Gamma} \boldsymbol{A} \xrightarrow{H} \boldsymbol{B}$ in $\boldsymbol{B}$. In this situation the colimit for Γ in $\boldsymbol{A}$ is obtained as follows; take the colimit cocone $\langle H\Gamma I \xrightarrow{i_I} \varinjlim H\Gamma\rangle_I$ in $\boldsymbol{B}$, this gives a cocone

$$\langle H\Gamma I \xrightarrow{H\theta\Gamma I^{-1}} HRH\Gamma I \xrightarrow{RHi_I} HR\varinjlim H\Gamma\rangle_I.$$

This induces an arrow $\gamma : \varinjlim H\Gamma \to HR\varinjlim H\Gamma$ such that for every I the diagram

$$\begin{tikzcd} H\Gamma I \arrow[rr, "i_I"] \arrow[dr, "HRi_I \circ H\theta\Gamma I^{-1}"'] & & \varinjlim H\Gamma \arrow[dl, "\gamma"] \\ & HR\varinjlim H\Gamma & \end{tikzcd} \tag{4.4}$$

commutes. Then $\theta R(\varinjlim H\Gamma) \circ R\gamma$ is an idempotent and a splitting of it produces the colimit of Γ in $\boldsymbol{A}$.

Remark 4.3. As a consequence of Theorem 4.6 we obtain that if a category is a retract of a complete category (in the sense that θ in 4.2 is the identity) then it is complete. This result appears in [7]

4.4 Pretoposes Revisited

We know from 1.6 that we have a 2-adjunction $\boldsymbol{Pretop} \underset{U}{\overset{F}{\rightleftarrows}} \boldsymbol{Lex}$. Denote by $\boldsymbol{T}$ the generated 2-monad. We use the results of the previous section to show that if a left exact category $\boldsymbol{C}$ has a $\boldsymbol{T}$-algebra structure then $\boldsymbol{C}$ is necesarily a pretopos.

Recall that for any $H : \boldsymbol{C} \to \boldsymbol{D}$ in $\boldsymbol{Lex}$, $F\boldsymbol{C} = (\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh}$ and $F(H) = Lan_{H^{op}}$. Let $\boldsymbol{T} = (T, \eta, \mu)$ be the 2-monad generated by $F \dashv U$.

If we start with an $\boldsymbol{T}$-algebra $(\boldsymbol{C}, \Phi)$ we have the following commutative diagram

$$\begin{tikzcd} \boldsymbol{C} \arrow[rr, "\eta\boldsymbol{C}"] \arrow[dr, "1_{\boldsymbol{C}}"'] & & T\boldsymbol{C} = (\boldsymbol{Set}^{\boldsymbol{C}^{op}})_{coh} \arrow[dl, "\Phi"] \\ & \boldsymbol{C} & \end{tikzcd}$$

Remember that $\eta\boldsymbol{C}$ is the factorization of the Yoneda embedding through $T\boldsymbol{C}$ and since $\boldsymbol{C}$ has split idempotents, we have by Theorem 4.6 that $\boldsymbol{C}$ has colimits of all those diagrams $\Gamma : \boldsymbol{I} \to \boldsymbol{C}$ for which the diagram $\boldsymbol{I} \xrightarrow{\Gamma} \boldsymbol{C} \xrightarrow{\eta\boldsymbol{C}} T\boldsymbol{C}$ has a colimit in $T\boldsymbol{C}$. It follows that $\boldsymbol{C}$ has initial object, finite coproducts and coequalizers of equivalence relations (equivalence relations are preserved by $\eta\boldsymbol{C}$ as it is left exact).

Proposition 4.7. *If $(\boldsymbol{C}, \Phi)$ is a $\boldsymbol{T}$-algebra then the initial object in $\boldsymbol{C}$ is strict.*

Proof. Denote by $\mathbf{0}$ the initial object of TC and by $\mathbb{O}$ the initial object of $TTC = (\boldsymbol{Set}^{((\boldsymbol{Set}^{C^{op}})_{coh})^{op}})_{coh}$. Following the image of the unique arrow $\mathbb{O} \to TC(\_,\mathbf{0})$ around the commutative diagram

$$\begin{tikzcd}
(\boldsymbol{Set}^{((\boldsymbol{Set}^{C^{op}})_{coh})^{op}})_{coh} \arrow[r, "\mu C"] \arrow[d, "Lan_{\Phi^{op}}"'] & (\boldsymbol{Set}^{C^{op}})_{coh} \arrow[d, "\Phi"] \\
(\boldsymbol{Set}^{C^{op}})_{coh} \arrow[r, "\Phi"'] & \boldsymbol{C}
\end{tikzcd} \tag{4.5}$$

we have on the one hand that $\Phi(\mu C(\mathbb{O} \to TC(\_,\mathbf{0}))) = \Phi(1_{\mathbf{0}}) = 1_{\Phi\mathbf{0}}$, and on the other $\Phi(Lan_{\Phi^{op}}(\mathbb{O} \to TC(\_,\mathbf{0})) = \Phi(\gamma)$ where $\gamma : \mathbf{0} \to \boldsymbol{C}(\_,\Phi\mathbf{0})$ is the unique morphism from $\mathbf{0}$ to $\boldsymbol{C}(\_,\Phi\mathbf{0})$. Since the initial object in $\boldsymbol{C}$ is obtained as a splitting of $\Phi(\gamma)$ we conclude that the initial object in $\boldsymbol{C}$ is $\Phi\mathbf{0}$. Given any arrow $f : D \to C$ in $\boldsymbol{C}$ we have that the square

$$\begin{tikzcd}
\mathbf{0} \arrow[r] \arrow[d] & \mathbf{0} \arrow[d] \\
\boldsymbol{C}(_,D) \arrow[r, "\boldsymbol{C}(_,f)"'] & \boldsymbol{C}(_,C)
\end{tikzcd}$$

is a pullback. Applying Φ we obtain the pullback

$$\begin{tikzcd}
\Phi\mathbf{0} \arrow[r] \arrow[d] & \Phi\mathbf{0} \arrow[d] \\
D \arrow[r, "f"'] & C
\end{tikzcd}$$

Therefore the initial object of $\boldsymbol{C}$ is stable under pullback. This means that the initial object is strict. $\square$

Proposition 4.8. *If $(\boldsymbol{C},\Phi)$ is a $\boldsymbol{T}$-algebra then finite coproducts in $\boldsymbol{C}$ are disjoint and stable.*

Proof. We do it for binary coproducts. Let C, D be objects of $\boldsymbol{C}$. Consider the arrow

$$TC(\_,\boldsymbol{C}(\_,C)) + TC(\_,\boldsymbol{C}(\_,D)) \xrightarrow{\langle TC(\_,i_1), TC(\_,i_2)\rangle} TC(\_,\boldsymbol{C}(\_,C) + \boldsymbol{C}(\_,D))$$

in $TT\boldsymbol{C}$ where $i_1 : \boldsymbol{C}(\_, C) \to \boldsymbol{C}(\_, C) + \boldsymbol{C}(\_, D)$ and $i_2 : \boldsymbol{C}(\_, D) \to \boldsymbol{C}(\_, C) + \boldsymbol{C}(\_, D)$ are the correspondig injections. We chase the arrow $\langle T\boldsymbol{C}(\_, i_1), T\boldsymbol{C}(\_, i_2)\rangle$ around the diagram 4.5. We obtain $\Phi(\mu\boldsymbol{C}(\langle T\boldsymbol{C}(\_, i_1), T\boldsymbol{C}(\_, i_2)\rangle)) = \Phi(1_{\boldsymbol{C}(\_,C)+\boldsymbol{C}(\_,D)}) = 1_{\Phi(\boldsymbol{C}(\_,C)+\boldsymbol{C}(\_,D))}$ on the one hand, and

$$\Phi(Lan_{\Phi^{op}}(\langle T\boldsymbol{C}(\_, i_1), T\boldsymbol{C}(\_, i_2)\rangle)) = \Phi(\langle \Phi(i_1), \Phi(i_2)\rangle)$$

on the other. Since the coproduct in $\boldsymbol{C}$ is obtained as a spliting of the idempotent $\Phi(\langle \Phi(i_1), \Phi(i_2)\rangle)$ we have that $\Phi(\boldsymbol{C}(\_, C) + \boldsymbol{C}(\_, D))$ is the coproduct in $\boldsymbol{C}$ of C and D. In other words $\Phi(\boldsymbol{C}(\_, C) + \boldsymbol{C}(\_, D)) = C + D$. We have that the square

$$\begin{tikzcd}
\mathbf{0} \arrow[r] \arrow[d] & \boldsymbol{C}(_, D) \arrow[d, "i_2"] \\
\boldsymbol{C}(_, C) \arrow[r, "i_1"'] & \boldsymbol{C}(_, C) + \boldsymbol{C}(_, D)
\end{tikzcd}$$

is a pullback. Applying Φ we get the pullback

$$\begin{tikzcd}
\Phi\mathbf{0} \arrow[r] \arrow[d] & D \arrow[d, "\Phi i_2"] \\
C \arrow[r, "\Phi i_1"'] & C + D
\end{tikzcd}$$

That is, the coproducts in $\boldsymbol{C}$ are disjoint.

For stability we use Lemma 1.6. Suppose we have $C + D \xrightarrow{\langle f_1, f_2\rangle} B \xleftarrow{g} A$ in $\boldsymbol{C}$. Then we have the pullback

$$\begin{tikzcd}
\boldsymbol{C}(_, P_1) + \boldsymbol{C}(_, P_2) \arrow[r, "\boldsymbol{C}(_, \pi_{12}) + \boldsymbol{C}(_, \pi_{22})"] \arrow[d, "\langle \boldsymbol{C}(_, \pi_{11}), \boldsymbol{C}(_, \pi_{21})\rangle"'] & \boldsymbol{C}(_, C) + \boldsymbol{C}(_, D) \arrow[d, "\langle \boldsymbol{C}(_, f_1), \boldsymbol{C}(_, f_2)\rangle"] \\
\boldsymbol{C}(_, A) \arrow[r, "\boldsymbol{C}(_, g)"'] & \boldsymbol{C}(_, B)
\end{tikzcd}$$

where the squares

$$\begin{tikzcd}
P_1 \arrow[r, "\pi_{12}"] \arrow[d, "\pi_{11}"'] & C \arrow[d, "f_1"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
P_2 \arrow[r, "\pi_{22}"] \arrow[d, "\pi_{21}"'] & D \arrow[d, "f_2"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

are pullbacks. Applying Φ we get the pullback

$$\begin{tikzcd}
P_1 + P_2 \arrow[r, "\pi_{12} + \pi_{22}"] \arrow[d, "\langle \pi_{11}, \pi_{21} \rangle"'] & C + D \arrow[d, "\langle f_1, f_2 \rangle"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

Therefore finite coproducts are stable in $\boldsymbol{C}$. $\square$

Proposition 4.9. *If $(\boldsymbol{C}, \Phi)$ is a $\boldsymbol{T}$-algebra then $\boldsymbol{C}$ has stable quotients of equivalence relations.*

Proof. Let $R \overset{r_1}{\underset{r_2}{\rightrightarrows}} C$ be an equivalence relation in $\boldsymbol{C}$, consider the quotient

$$\boldsymbol{C}(\_, R) \overset{\boldsymbol{C}(\_, r_1)}{\underset{\boldsymbol{C}(\_, r_2)}{\rightrightarrows}} \boldsymbol{C}(\_, C) \xrightarrow{q} Q$$

in $T\boldsymbol{C}$ and the quotient

$$T\boldsymbol{C}(\_, \boldsymbol{C}(\_, R)) \overset{T\boldsymbol{C}(\_, \boldsymbol{C}(\_, r_1))}{\underset{T\boldsymbol{C}(\_, \boldsymbol{C}(\_, r_2))}{\rightrightarrows}} T\boldsymbol{C}(\_, \boldsymbol{C}(\_, C)) \xrightarrow{\chi} \mathcal{Q}$$

in $TT\boldsymbol{C}$. There exists then a unique arrow $t : \mathcal{Q} \to T\boldsymbol{C}(\_, Q)$ such that the diagram

$$\begin{tikzcd}
T\boldsymbol{C}(_, \boldsymbol{C}(_, C)) \arrow[rr, "\chi"] \arrow[dr, "T\boldsymbol{C}(_, q)"'] & & \mathcal{Q} \arrow[dl, "t"] \\
& T\boldsymbol{C}(_, Q) &
\end{tikzcd}$$

commutes. It is easy to see that $\mu\boldsymbol{C}(t)$ is an isomorphism and therefore $\Phi(\mu\boldsymbol{C}(t))$ is an isomorphism. On the other hand we have that $\Phi(Lan_{\Phi^{op}}(t)) = \Phi(Q \xrightarrow{\gamma} \boldsymbol{C}(\_, \Phi Q))$ where γ is the uniqu arrow that makes the diagram

$$\begin{tikzcd}
\boldsymbol{C}(_, C) \arrow[rr, "q"] \arrow[dr, "\boldsymbol{C}(_, \Phi q)"'] & & Q \arrow[dl, "\gamma"] \\
& \boldsymbol{C}(_, \Phi Q) &
\end{tikzcd}$$

commute. Since the coequalizer of (r_1, r_2) in $\boldsymbol{C}$ is obtained by splitting $\Phi(\gamma)$, we have that the coequalizer is $\Phi(q) : C \to \Phi(Q)$. Since the square

$$\begin{tikzcd}
\boldsymbol{C}(_, R) \arrow[r, "\boldsymbol{C}(_, r_1)"] \arrow[d, "\boldsymbol{C}(_, r_2)"'] & \boldsymbol{C}(_, C) \arrow[d, "q"] \\
\boldsymbol{C}(_, C) \arrow[r, "q"'] & Q
\end{tikzcd}$$

is a pullback, applying Φ we get the pullback

$$\begin{tikzcd}
R \arrow[r, "r_1"] \arrow[d, "r_2"'] & C \arrow[d, "\Phi q"] \\
C \arrow[r, "\Phi q"'] & \Phi Q
\end{tikzcd}$$

That is, Φq is a quotient in $\boldsymbol{C}$ of the equivalence relation (r_1, r_2). We show that Φq is stable. Suppose we have an arrow $g : D \to \Phi Q$ in $\boldsymbol{C}$. Consider the pullback

$$\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "\Phi q"] \\
D \arrow[r, "g"'] & \Phi Q
\end{tikzcd}$$

in $\boldsymbol{C}$, and the pullback

$$\begin{tikzcd}
U \arrow[r, "u_2"] \arrow[d, "u_1"'] & Q \arrow[d, "\gamma"] \\
\boldsymbol{C}(_, D) \arrow[r, "\boldsymbol{C}(_, g)"'] & \boldsymbol{C}(_, \Phi Q)
\end{tikzcd} \tag{4.6}$$

in $T\boldsymbol{C}$. There exists a unique arrow $q' : \boldsymbol{C}(\_, P) \to U$ such that the diagram

$$\begin{tikzcd}
 & \boldsymbol{C}(_, P) \arrow[r, "\boldsymbol{C}(_, \pi_2)"] \arrow[dl, "\boldsymbol{C}(_, \pi_1)"'] \arrow[d, "q'"] & \boldsymbol{C}(_, C) \arrow[d, "q"] \\
\boldsymbol{C}(_, D) & U \arrow[l, "u_1"] \arrow[r, "u_2"'] & Q
\end{tikzcd}$$

commutes. Since the diagrams above involving P and U are pullbacks it can be shown that the square on the right in the previous diagram is a pullback. Consider the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & S_2 \arrow[r] \arrow[d] & P \arrow[d, "\pi_2"] \\
S_1 \arrow[r] \arrow[d] & R \arrow[r, "r_2"'] \arrow[d, "r_1"] & C \\
P \arrow[r, "r_2"'] & C &
\end{tikzcd}$$

in which every square is a pullback. Since the inner square in the commutative diagram

$$\begin{tikzcd}
\boldsymbol{C}(_,S) \arrow[rrr] \arrow[dr] \arrow[ddd] & & & \boldsymbol{C}(_,P) \arrow[dl, "\boldsymbol{C}(_,\pi_2)"'] \arrow[ddd, "q'"] \\
& \boldsymbol{C}(_,R) \arrow[r, "\boldsymbol{C}(_,r_1)"] \arrow[d, "\boldsymbol{C}(_,r_1)"'] & \boldsymbol{C}(_,C) \arrow[d, "q"] & \\
& \boldsymbol{C}(_,C) \arrow[r, "q"'] & Q & \\
\boldsymbol{C}(_,P) \arrow[ur, "\boldsymbol{C}(_,\pi_2)"'] \arrow[rrr, "q'"'] & & & U \arrow[ul, "u_2"']
\end{tikzcd}$$

is a pullback it is not hard to see that the outer square is also a pullback. Therefore the kernel pair of q' is $\boldsymbol{C}(\_,S) \rightrightarrows \boldsymbol{C}(\_,P)$. Since quotients of equivalence relations are stable in $T\boldsymbol{C}$ and q' is the pullback of q along u_2 we have that the diagram $\boldsymbol{C}(\_,S) \rightrightarrows \boldsymbol{C}(\_,P) \xrightarrow{q'} U$ is a quotient diagram. Therefore $P \xrightarrow{\Phi q'} \Phi U$ is the quotient of the equivalence relation $S \rightrightarrows P$ in $\boldsymbol{C}$. $\square$

As a corollary we have

Proposition 4.10. *If $(\boldsymbol{C}, \Phi)$ is a $\boldsymbol{T}$-algebra then $\boldsymbol{C}$ is a pretopos.* $\square$

Similarly we can show

Proposition 4.11. *If $(F,\varphi) : (\boldsymbol{C},\Phi) \to (\boldsymbol{D},\Psi)$ is a $\boldsymbol{T}$-ALG morphism then F is an elementary functor.* $\square$

4.5 2-Algebras Over CAT

4.5.1 CAT over CAT

Consider the 2-adjunction

$$\begin{tikzcd} \boldsymbol{CAT}^{op} \arrow[d, shift left, "\boldsymbol{Set}^{(-)}"] \\ \boldsymbol{CAT} \arrow[u, shift left, "\boldsymbol{Set}^{(-)}"] \end{tikzcd}$$

whose unit $\eta\boldsymbol{A} : \boldsymbol{A} \to \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})$ is evaluation, that is $\eta\boldsymbol{A}(A) = ev_A$ and $\eta\boldsymbol{A}(a) = ev_a$ for every $a : A \to A'$ in $\boldsymbol{A}$, and whose counit $\varepsilon\boldsymbol{B} : \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{B}}, \boldsymbol{Set}) \to \boldsymbol{B}$ in $\boldsymbol{CAT}^{op}$ is also the evaluation $\boldsymbol{B} \to \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{B}}, \boldsymbol{Set})$. We consider the 2-monad $\boldsymbol{T} = (T, \eta, \mu)$ generated by the 2-adjunction above. We have that

$$\mu\boldsymbol{A} : \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}},\boldsymbol{Set})}, \boldsymbol{Set}) \to \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})$$

is such that $\mu\boldsymbol{A}(\mathcal{L})(G) = \mathcal{L}(ev_G)$, $\mu\boldsymbol{A}(\mathcal{L})(\sigma) = \mathcal{L}(ev_\sigma)$ and $\mu\boldsymbol{A}(h)(G) = hev_G$ for every $h : \mathcal{L} \to \mathcal{L}'$ in $\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}},\boldsymbol{Set})}, \boldsymbol{Set})$ and every $\sigma : G \to G'$ in $\boldsymbol{Set}^{\boldsymbol{A}}$.

Given a diagram $\Gamma : \boldsymbol{I} \to \boldsymbol{A}$ we will denote the composition

$$\boldsymbol{I} \xrightarrow{\Gamma} \boldsymbol{A} \xrightarrow{\eta\boldsymbol{A}} \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})$$

by ev_Γ.

Proposition 4.12. *If $(\boldsymbol{A},\Phi)$ is a strict $\boldsymbol{T}$-algebra then $\boldsymbol{A}$ is a complete and cocomplete category and Φ preserves limits and colimits of diagrams of the form ev_Γ with $\Gamma : \boldsymbol{I} \to \boldsymbol{A}$.*

Proof. We have the commutative diagram

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "\eta\boldsymbol{A}"] \arrow[dr, "1_{\boldsymbol{A}}"'] & \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set}) \arrow[d, "\Phi"] \\ & \boldsymbol{A} \end{tikzcd}$$

Now, $\boldsymbol{A}$ has split idempotents (see Remark 4.2) and by Theorem 4.6 we have that $\boldsymbol{A}$ is complete and cocomplete. Let $\Gamma : \boldsymbol{I} \to \boldsymbol{A}$ be a diagram. To obtain the limit for Γ in $\boldsymbol{A}$ we have to proceed as follows according to Proposition 4.6: First we consider the limit cone $\langle \varprojlim_I ev_\Gamma \xrightarrow{\pi_I} ev_{\Gamma I}\rangle_I$ in $\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})$. To this we apply Φ and we get a cone $\langle \Phi(\varprojlim_I ev_\Gamma) \xrightarrow{\Phi\pi_I} \Gamma I\rangle_I$ in $\boldsymbol{A}$. From this one we obtain the cone $\langle ev_{(\varprojlim ev_\Gamma)}(G) \xrightarrow{ev_{\Phi\pi_I}} ev_{\Gamma I}\rangle_I$ in $\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})$. There exists then a unique arrow $\gamma : ev_{\Phi(\varprojlim ev_\Gamma)} \to \varprojlim ev_\Gamma$ such that for every I in $\boldsymbol{I}$ the diagram

$$\begin{tikzcd}
ev_{\Phi(\varprojlim ev_\Gamma)} \arrow[rr, "ev_{\Phi\pi_I}"] \arrow[dr, "\gamma"'] & & ev_{\Phi ev_{\Gamma I}} = ev_{\Gamma I} \\
& \varprojlim ev_\Gamma \arrow[ur, "\pi_I"'] &
\end{tikzcd}$$

commutes (compare with 4.4). We have that $\Phi\gamma : \Phi\varprojlim_I ev_\Gamma \to \Phi\varprojlim_I ev_\Gamma$ is an idempotent and the limit of Γ in $\boldsymbol{A}$ is obtained by splitting $\Phi\gamma$. It is enough then to show that $\Phi\gamma$ is an isomorphism. To do this consider the unique arrow $\zeta : ev_{(\varprojlim ev_\Gamma)} \to \varprojlim ev_{ev_\Gamma}$ in $\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})}, \boldsymbol{Set})$ that makes the diagram

$$\begin{tikzcd}
ev_{(\varprojlim ev_\Gamma)} \arrow[rr, "ev_{\pi_I}"] \arrow[dr, "\zeta"'] & & ev_{ev_{\Gamma I}} \\
& \varprojlim ev_{ev_\Gamma} \arrow[ur, "\pi'_I"'] &
\end{tikzcd} \tag{4.7}$$

commute. We chase ζ around the commutative diagram

$$\begin{tikzcd}
\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})}, \boldsymbol{Set}) \arrow[r, "\boldsymbol{CAT}(\boldsymbol{Set}^{\Phi}, \boldsymbol{Set})"] \arrow[d, "\mu\boldsymbol{A}"'] & \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set}) \arrow[d, "\Phi"] \\
\boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set}) \arrow[r, "\Phi"'] & \boldsymbol{A}.
\end{tikzcd} \tag{4.8}$$

Observe that if $G : \boldsymbol{A} \to \boldsymbol{Set}$ we have

$$\begin{aligned}
\boldsymbol{CAT}(\boldsymbol{Set}^{\Phi}, \boldsymbol{Set})(ev_{(\varprojlim ev_\Gamma)})(G) &= ev_{(\varprojlim ev_\Gamma)} \circ \boldsymbol{Set}^{\Phi}(G) \\
&= ev_{(\varprojlim ev_\Gamma)}(G \circ \Phi) \\
&= G(\Phi(\varprojlim ev_\Gamma)) \\
&= (ev_{\Phi(\varprojlim ev_\Gamma)})(G)
\end{aligned}$$

Similarly we have that

$$\boldsymbol{CAT}(\boldsymbol{Set}^{\Phi}, \boldsymbol{Set})(\varprojlim ev_{ev_\Gamma}) = \varprojlim ev_\Gamma$$

So applying $\boldsymbol{CAT}(\boldsymbol{Set}^{\Phi}, \boldsymbol{Set})$ to diagram 4.7 we obtain the commutative diagram

$$\begin{tikzcd}
ev_{\Phi(\varprojlim ev_\Gamma)} \arrow[rr, "ev_{\pi_I}"] \arrow[dr, "\boldsymbol{CAT}(\boldsymbol{Set}^{\Phi}, \boldsymbol{Set})(\zeta)"'] & & ev_{\Phi ev_\Gamma} \\
& \varprojlim ev_\Gamma \arrow[ur, "\pi_I"'] &
\end{tikzcd}$$

That is $\boldsymbol{CAT}(\boldsymbol{Set}^{\Phi}, \boldsymbol{Set})(\zeta) = \gamma$. Therefore $\Phi(\boldsymbol{CAT}(\boldsymbol{Set}^{\Phi}, \boldsymbol{Set})(\zeta)) = \Phi(\gamma)$. On the other hand it is not hard to see that $\mu\boldsymbol{A}(\zeta) = 1_{(\varprojlim ev_\Gamma)}$ and therefore $\Phi(\mu\boldsymbol{A}(\zeta)) = 1_{\Phi(\varprojlim ev_\Gamma)}$. That is $\Phi(\gamma) = 1_{\Phi(\varprojlim ev_\Gamma)}$. □

Proposition 4.13. *If* $(H, \varphi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{B}, \Psi)$ *is a morphism of* $\boldsymbol{T}$*-algebras then* $H : \boldsymbol{A} \to \boldsymbol{B}$ *preserves limits and colimits.*

Proof. Let $\boldsymbol{I}$ be a small category. Consider

$$\begin{tikzcd}
\boldsymbol{A}^{\boldsymbol{I}} \arrow[r, "\eta\boldsymbol{A}^{\boldsymbol{I}}"] \arrow[d, "H^{\boldsymbol{I}}"'] & \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})^{\boldsymbol{I}} \arrow[r, "\varprojlim"] \arrow[d, "\boldsymbol{CAT}(\boldsymbol{Set}^{H}, \boldsymbol{Set})^{\boldsymbol{I}}"] & \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set}) \arrow[r, "\Phi"] \arrow[d, "\boldsymbol{CAT}(\boldsymbol{Set}^{H}, \boldsymbol{Set})"] & \boldsymbol{A} \arrow[d, "H"] \arrow[dl, phantom, "\swarrow\varphi"] \\
\boldsymbol{B}^{\boldsymbol{I}} \arrow[r, "\eta\boldsymbol{B}^{\boldsymbol{I}}"'] & \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{B}}, \boldsymbol{Set})^{\boldsymbol{I}} \arrow[r, "\varprojlim"'] & \boldsymbol{CAT}(\boldsymbol{Set}^{\boldsymbol{B}}, \boldsymbol{Set}) \arrow[r, "\Psi"'] & \boldsymbol{B}
\end{tikzcd}$$

It is easy to see that the middle and left squares above commute. Given $\Gamma : \boldsymbol{I} \to \boldsymbol{A}$ we obtain with the help of the coherence diagrams the commutative diagram

$$\begin{tikzcd}
H(\Phi(\varprojlim ev_\Gamma)) \arrow[rr, "H\Phi\pi_I"] \arrow[dr, "\varphi \varprojlim ev_\Gamma"'] & & H\Gamma I \\
& \Psi(\varprojlim ev_{H\Gamma}) \arrow[ur, "\Psi\pi_I"'] &
\end{tikzcd}$$

Colimits are done the same way. □

Notice that φ above gives the isomorphisms $\varphi\varprojlim ev_\Gamma : H(\varprojlim \Gamma) \to \varprojlim H\Gamma$ and $(\varphi\varinjlim ev_\Gamma)^{-1} : \varinjlim H\Gamma \to H(\varinjlim \Gamma)$ induced by the universal property of $\varprojlim$ and $\varinjlim$ on $\boldsymbol{B}$.

4.5.2 $\boldsymbol{LEX}$ over CAT

Similarly we can consider the 2-adjunction

$$\begin{tikzcd} \boldsymbol{LEX}^{op} \arrow[d, shift left, "\boldsymbol{LEX}(\_,\boldsymbol{Set})"] \\ CAT \arrow[u, shift left, "\boldsymbol{Set}^{(-)}"] \end{tikzcd}$$

and carry over the same argument. We obtain a 2-monad that we (also) denote by $\boldsymbol{T} = (T, \eta, \mu)$. The corresponding proposition is

Proposition 4.14. *If $(\boldsymbol{A}, \Phi)$ is a $\boldsymbol{T}$-algebra then $\boldsymbol{A}$ has all limits and filtered colimits. Furthermore Φ preserves limits of the form $\boldsymbol{I} \xrightarrow{\Gamma} \boldsymbol{A} \xrightarrow{\eta \boldsymbol{A}} \boldsymbol{LEX}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})$ and colimits of the form $\boldsymbol{J} \xrightarrow{\Theta} \boldsymbol{A} \xrightarrow{\eta \boldsymbol{A}} \boldsymbol{LEX}(\boldsymbol{Set}^{\boldsymbol{A}}, \boldsymbol{Set})$ where $\boldsymbol{J}$ is filtered. If $(H, \varphi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{B}, \Psi)$ is a morphism of $\boldsymbol{T}$-algebras then H preserves limits and filtered colimits.* $\square$

4.5.3 $\boldsymbol{PRETOP}$ over CAT

Consider now the 2-adjunction

$$\begin{tikzcd} \boldsymbol{PRETOP}^{op} \arrow[d, shift left, "\boldsymbol{Mod}(\_)"] \\ CAT \arrow[u, shift left, "\boldsymbol{Set}^{(-)}"] \end{tikzcd}$$

and the generated 2-monad $\boldsymbol{T} = (T, \eta, \mu)$. We have

Proposition 4.15. *If $(\boldsymbol{A}, \Phi)$ is a $\boldsymbol{T}$-algebra then $\boldsymbol{A}$ has filtered colimits and Φ preserves colimits of the form $\boldsymbol{I} \xrightarrow{\Gamma} \boldsymbol{A} \xrightarrow{\eta \boldsymbol{A}} \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}})$. If $(H, \varphi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{B}, \Psi)$ is a morphism of $\boldsymbol{T}$-algebras then H preserves filtered colimits.* $\square$

It is to be expected that in this setting we can give a pre-ultracategory structure to any $\boldsymbol{T}$-algebra $(\boldsymbol{A}, \Phi)$ in much the same way as we have constructed limits and colimits up to here. This is what we do now.

We define the 2-functor $W : \boldsymbol{T}\text{-}ALG \to \boldsymbol{PUC}$ as follows. Given $(\boldsymbol{A}, \Phi)$ in $\boldsymbol{T}\text{-}ALG$ then the underlying category of $W(\boldsymbol{A}, \Phi)$ is $\boldsymbol{A}$ and given an ultrafilter $(I, \mathcal{U})$ define $[\mathcal{U}]_{W(\boldsymbol{A},\Phi)} : \boldsymbol{A}^I \to \boldsymbol{A}$ as the composition

$$\boldsymbol{A}^I \xrightarrow{(\eta \boldsymbol{A})^I} \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}})^I \xrightarrow{[\mathcal{U}]} \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}}) \xrightarrow{\Phi} \boldsymbol{A}.$$

where $[\mathcal{U}]$ denotes the usual ultraproduct functor of models. If $(H, \varphi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{B}, \Psi)$ is a morphism of $\boldsymbol{T}$-algebras, then we define $W(H, \varphi) = H$ together with the natural isomorphisms

$$\begin{tikzcd}
\boldsymbol{A}^I \arrow[r, "{[\mathcal{U}]_{W(\boldsymbol{A},\Phi)}}"] \arrow[d, "H^I"'] & \boldsymbol{A} \arrow[d, "H"] \arrow[dl, Rightarrow, "\varphi[\mathcal{U}](\eta \boldsymbol{A})^I"'] \\
\boldsymbol{B}^I \arrow[r, "{[\mathcal{U}]_{W(\boldsymbol{B}.\Psi)}}"'] & \boldsymbol{B}.
\end{tikzcd}$$

The natural isomorphism $\varphi[\mathcal{U}](\eta \boldsymbol{A})^I$ has the domain and codomain shown above due to the fact that the diagram

$$\begin{tikzcd}
\boldsymbol{A}^I \arrow[r, "(\eta \boldsymbol{A})^I"] \arrow[d, "H^I"'] & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}})^I \arrow[r, "{[\mathcal{U}]}"] \arrow[d, "(\boldsymbol{Mod}(\boldsymbol{Set}^H))^I"] & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}}) \arrow[d, "\boldsymbol{Mod}(\boldsymbol{Set}^H)"] \\
\boldsymbol{B}^I \arrow[r, "(\eta \boldsymbol{B})^I"] & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{B}})^I \arrow[r, "{[\mathcal{U}]}"] & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{B}})
\end{tikzcd}$$

commutes on the nose. If $\tau : (H, \varphi) \to (K, \psi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{B}, \Psi)$ is in $\boldsymbol{T}\text{-}ALG$ define $W(\tau) = \tau : H \to K$. We have to show that τ is a pre-ultranatural transformation. It is easy to see that

$$\begin{tikzcd}
\boldsymbol{A} \arrow[r, "(\eta \boldsymbol{A})^I"] & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}})^I \arrow[r, bend left, "\boldsymbol{Mod}(\boldsymbol{Set}^H)^I"] \arrow[r, bend right, "\boldsymbol{Mod}(\boldsymbol{Set}^K)^I"'] \arrow[r, phantom, "\Downarrow \boldsymbol{Mod}(\boldsymbol{Set}^\tau)^I"] & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{B}})^I
\end{tikzcd}$$

equals

$$\begin{tikzcd}
\boldsymbol{A}^I \arrow[r, bend left, "H^I"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow \tau^I"] & \boldsymbol{B}^I \arrow[r, "(\eta \boldsymbol{B})^I"] & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{B}})^I
\end{tikzcd}$$

and that

$$\begin{tikzcd}[column sep=large]
\mathbf{Mod}(\mathbf{Set}^{\mathbf{A}})^I \arrow[r, "\mathbf{Mod}(\mathbf{Set}^H)^I"] & \mathbf{Mod}(\mathbf{Set}^{\mathbf{A}}) \arrow[rr, bend left, "\mathbf{Mod}(\mathbf{Set}^H)"] \arrow[rr, bend right, "\mathbf{Mod}(\mathbf{Set}^K)"'] \arrow[rr, phantom, "\Downarrow \mathbf{Mod}(\mathbf{Set}^\tau)"] & & \mathbf{Mod}(\mathbf{Set}^{\mathbf{B}})
\end{tikzcd}$$

equals

$$\begin{tikzcd}[column sep=large]
\mathbf{Mod}(\mathbf{Set}^{\mathbf{A}})^I \arrow[rr, bend left, "\mathbf{Mod}(\mathbf{Set}^H)^I"] \arrow[rr, bend right, "\mathbf{Mod}(\mathbf{Set}^K)^I"'] \arrow[rr, phantom, "\Downarrow \mathbf{Mod}(\mathbf{Set}^\tau)^I"] & & \mathbf{Mod}(\mathbf{Set}^{\mathbf{B}})^I \arrow[r, "{[\mathcal{U}]}"] & \mathbf{Mod}(\mathbf{Set}^{\mathbf{B}})
\end{tikzcd}$$

Since τ is a 2-cell in $\boldsymbol{T}$-ALG we also have that

$$\begin{tikzcd}[column sep=large]
\mathbf{Mod}(\mathbf{Set}^{\mathbf{A}}) \arrow[rr, bend left, "\mathbf{Mod}(\mathbf{Set}^H)"] \arrow[rr, bend right, "\mathbf{Mod}(\mathbf{Set}^K)"'] \arrow[rr, phantom, "\Uparrow \mathbf{Mod}(\mathbf{Set}^\tau)"] \arrow[d, "\Phi"'] & & \mathbf{Mod}(\mathbf{Set}^{\mathbf{B}}) \arrow[d, "\Psi"] \\
\mathbf{A} \arrow[rr, "H"'] \arrow[urr, phantom, "\nearrow \varphi"] & & \mathbf{B}
\end{tikzcd}$$

equals

$$\begin{tikzcd}[column sep=large]
\mathbf{Mod}(\mathbf{Set}^{\mathbf{A}}) \arrow[r, "\mathbf{Mod}(\mathbf{Set}^K)"] \arrow[d, "\Phi"'] & \mathbf{Mod}(\mathbf{Set}^{\mathbf{B}}) \arrow[d, "\Psi"] \\
\mathbf{A} \arrow[r, bend left, "K"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Uparrow \tau"] \arrow[ur, phantom, "\nearrow \psi"] & \mathbf{B}
\end{tikzcd}$$

It follows that

$$\begin{tikzcd}
\mathbf{A}^I \arrow[r, bend left, "K^I"] \arrow[r, bend right, "H^I"'] \arrow[r, phantom, "\Uparrow \tau^I"] \arrow[d, "\Phi"'] & \mathbf{B}^I \arrow[d, "\Psi"] \\
\mathbf{A} \arrow[r, "H"'] \arrow[ur, phantom, "\nearrow \varphi{[\mathcal{U}]}\eta\mathbf{A}^I"] & \mathbf{B}
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{A}^I \arrow[r, "K^I"] \arrow[d, "\Phi"'] & \mathbf{B}^I \arrow[d, "\Psi"] \\
\mathbf{A} \arrow[r, bend left, "K"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Uparrow \tau"] \arrow[ur, phantom, "\nearrow \psi{[\mathcal{U}]}\eta\mathbf{B}^I"] & \mathbf{B}
\end{tikzcd}$$

That is, τ is a 2-cell in $\boldsymbol{PUC}$. This completes the definition of the 2-functor W.

Given a pretopos $\boldsymbol{P}$ define $\Phi_{\boldsymbol{P}} : \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\boldsymbol{P})}) \to \boldsymbol{Mod}(\boldsymbol{P})$ such that

$$\Phi_{\boldsymbol{P}}(\mathcal{M})(P) = \mathcal{M}(ev_P)$$

for every $\mathcal{M}$ in $\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\boldsymbol{P})})$ and every P in $\boldsymbol{P}$. It is easy to see that $\Phi_{\boldsymbol{P}}(\mathcal{M})$ is an elementary functor. If $h : \mathcal{M} \to \mathcal{N}$ is a morphism in $\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\boldsymbol{P})})$ define $\Phi_{\boldsymbol{P}}(h)(P) = h(ev_P)$ for every $P \in \boldsymbol{P}$. Notice that the 2-adjunction

$$\begin{tikzcd} \boldsymbol{PRETOP}^{op} \arrow[d, shift left, "\boldsymbol{Mod}(\_)"] \\ \boldsymbol{CAT} \arrow[u, shift left, "\boldsymbol{Set}^{(-)}"] \end{tikzcd}$$

gives us the comparison 2-functor $\boldsymbol{PRETOP}^{op} \to (\boldsymbol{T}\text{-}ALG)_s$ and it is not hard to see that this functor is such that $\boldsymbol{P} \mapsto (\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}})$ for every pretopos $\boldsymbol{P}$. The 2-functor in the following definition is simply the comparison 2-functor $\boldsymbol{PRETOP} \to (\boldsymbol{T}\text{-}ALG)_s$ followed by the inclusion $(\boldsymbol{T}\text{-}ALG)_s \to \boldsymbol{T}\text{-}ALG$.

Definition 4.3. Let $(\boldsymbol{Mod}(\_), \Phi_{(\_)}) : \boldsymbol{PRETOP}^{op} \to \boldsymbol{T}\text{-}ALG$ be the 2-functor such that for every 2-cell

$$\begin{tikzcd} \boldsymbol{P} \arrow[r, bend left, "E"] \arrow[r, bend right, "E'"'] \arrow[r, phantom, "\downarrow\sigma"] & \boldsymbol{Q} \end{tikzcd}$$

in $\boldsymbol{PRETOP}$ we have that $(\boldsymbol{Mod}(\_), \Phi_{(\_)})$ applied to it gives

$$\begin{tikzcd} (\boldsymbol{Mod}(\boldsymbol{Q}), \Phi_{\boldsymbol{Q}}) \arrow[r, bend left, "{(\boldsymbol{Mod}(E), =)}"] \arrow[r, bend right, "{(\boldsymbol{Mod}(E'), =)}"'] \arrow[r, phantom, "\downarrow \boldsymbol{Mod}(\tau)"] & (\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}}) \end{tikzcd}$$

In particular when $\boldsymbol{P}$ is the full subcategory of $\boldsymbol{Set}^{\boldsymbol{Set}_0}$ whose objects are the finitely generated functors, where $\boldsymbol{Set}_0$ is the category of finite sets, we have that $\boldsymbol{Mod}(\boldsymbol{P})$ is equivalent to the category $\boldsymbol{Set}$ where the equivalence is given by $ev_{in} : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ where $in : \boldsymbol{Set}_0 \to \boldsymbol{Set}$ is the inclusion. It is not hard to see that $\Phi_{\boldsymbol{P}}$ defined above corresponds to the functor $\Psi_{\boldsymbol{Set}} : \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Set}}) \to \boldsymbol{Set}$ defined as $\Psi_{\boldsymbol{Set}}\mathcal{M} = \mathcal{M}(id_{\boldsymbol{Set}})$. This gives us the $\boldsymbol{T}$-algebra $(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})$.

Proposition 4.16. *The functor $W : \boldsymbol{T}\text{-}ALG \to \boldsymbol{PUC}$ defined above is such that*

$$W(\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}}) = \underline{\boldsymbol{Mod}}(\boldsymbol{P})$$

for any pretopos $\boldsymbol{P}$. In particular $W(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}}) = \underline{\boldsymbol{Set}}$.

Proof. Let $(I,\mathcal{U})$ be an ultrafilter then $[\mathcal{U}]_{W(\boldsymbol{Mod}(\boldsymbol{P}),\Phi_{\boldsymbol{P}})}$ is the compositon

$$\begin{tikzcd}
\boldsymbol{Mod}(\boldsymbol{P})^I \arrow[r, "\eta\boldsymbol{Mod}(\boldsymbol{P})^I"] & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\boldsymbol{P})})^I \arrow[d, "{[\mathcal{U}]}"] \\
\boldsymbol{Mod}(\boldsymbol{P}) & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\boldsymbol{P})}) \arrow[l, "\Phi_{\boldsymbol{P}}"']
\end{tikzcd}$$

If we start with a family $\langle M_i \rangle_I$ in $\boldsymbol{Mod}(\boldsymbol{P})^I$ we obtain the model $\Phi_{\boldsymbol{P}}(\prod_i ev_{M_i}/\mathcal{U})$ in $\boldsymbol{Mod}(\boldsymbol{P})$. For any P in $\boldsymbol{P}$ we have

$$\Phi_{\boldsymbol{P}}(\prod_i ev_{M_i}/\mathcal{U})(P) = \prod_i ev_{M_i}/\mathcal{U}(ev_P) = \prod_i ev_{M_i}(ev_P)/\mathcal{U} = \prod_i M_i P/\mathcal{U}.$$

Therefore $[\mathcal{U}]_{W(\boldsymbol{Mod}(\boldsymbol{P}),\Phi_{\boldsymbol{P}})} : \boldsymbol{Mod}(\boldsymbol{P})^I \to \boldsymbol{Mod}(\boldsymbol{P})$ is the usual ultraproduct functor.
□

In other words we have a commutative diagram of 2-functors

$$\begin{tikzcd}
\boldsymbol{PRETOP}^{op} \arrow[rr, "(\boldsymbol{Mod}(_), \Phi_{(_)})"] \arrow[dr, "\underline{\boldsymbol{Mod}}(_)"'] & & \boldsymbol{T}\text{-}ALG \arrow[dl, "W"] \\
& \boldsymbol{PUC} &
\end{tikzcd}$$

Proposition 4.17. *Given a morphism $(\boldsymbol{A}, \Phi) \xrightarrow{(H,\varphi)} (\boldsymbol{B}, \Psi)$ in $\boldsymbol{T}\text{-}ALG$ we have that the category $(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(\boldsymbol{A},\Phi)}$ is a pretopos and $(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(H,\varphi)}$ is an elementary functor. Furthermore, the corresponding limits and colimits are created by the forgetful functor $(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(\boldsymbol{A},\Phi)} \to \boldsymbol{Set}^{\boldsymbol{A}}$.*

Proof. We only do the finite limits to illustrate the point, the rest of the constructions are done similarly. Suppose $\Gamma : \boldsymbol{J} \to (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(\boldsymbol{A},\Phi)}$ is a diagram with $\boldsymbol{J}$ finite.

Denote the image of J under Γ by the pair $(\Gamma J, \gamma J)$. Then for any $\mathcal{M}$ in $\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}})$ we have $\gamma J(\mathcal{M}) : \Gamma J(\Phi\mathcal{M}) \to \mathcal{M}(\Gamma J)$. Consider the limit $\varprojlim_J \Gamma J$ in $\boldsymbol{Set}^{\boldsymbol{A}}$. We want a natural isomorphism γ

$$\begin{tikzcd}
\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}}) \arrow[r, "\Phi"] \arrow[d, "\boldsymbol{Mod}(\boldsymbol{Set}^{\varprojlim_J \Gamma J})"'] & \boldsymbol{A} \arrow[d, "\varprojlim_J \Gamma J"] \arrow[dl, phantom, "\swarrow\gamma"] \\
\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Set}}) \arrow[r, "\Psi_{\boldsymbol{Set}}"'] & \boldsymbol{Set}
\end{tikzcd}$$

Let $\mathcal{M}$ be an object of $\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}})$ let $\gamma\mathcal{M}$ be the unique arrow that makes the following diagram commute

$$\begin{tikzcd}
\varprojlim_J \Gamma J(\Phi\mathcal{M}) \arrow[rr, "\pi_J"] \arrow[d, "\gamma\mathcal{M}"'] & & \Gamma J(\Phi\mathcal{M}) \arrow[d, "\gamma J\mathcal{M}"] \\
\mathcal{M}(\varprojlim_J \Gamma J) \arrow[r, "\simeq"] & \varprojlim_J \mathcal{M}(\Gamma J) \arrow[r, "\pi_J"] & \mathcal{M}(\Gamma J)
\end{tikzcd}$$

for every J in $\boldsymbol{J}$, where the iso $\mathcal{M}(\varprojlim_J \Gamma J) \to \varprojlim_J \mathcal{M}(\Gamma J)$ comes from the fact that $\mathcal{M}$ is an elementary functor. It is not hard to see that γ is indeed natural, satisfies the coherence conditions and that $(\varprojlim_J \Gamma J, \gamma)$ is the limit of the diagram $\Gamma : \boldsymbol{J} \to (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(\boldsymbol{A},\Phi)}$. $\square$

We can then make the following definition

Definition 4.4. Let $\mathcal{P}$ denote the 2-functor

$$\mathcal{P} = \boldsymbol{T}\text{-}ALG(\_\_, (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})) : \boldsymbol{T}\text{-}ALG \to \boldsymbol{PRETOP}^{op}$$

We define now a new 2-monad $\boldsymbol{S} = (S, \xi, \nu)$, this time over $\boldsymbol{T}$-ALG.

In view of proposition 4.17 we can regard the category $\boldsymbol{Set}$ as a schizophrenic object in the categories $\boldsymbol{PRETOP}$ and $\boldsymbol{T}$-ALG. This gives rise to the 2-adjunction

$$\begin{tikzcd}
\boldsymbol{PRETOP}^{op} \arrow[d, shift left, "(\boldsymbol{Mod}(_), \Phi_{(_)})"] \\
\boldsymbol{T}\text{-}ALG \arrow[u, shift left, "\mathcal{P}"]
\end{tikzcd}$$

with unit $\xi : id_{\boldsymbol{T}\text{-}ALG} \to (\boldsymbol{Mod}(\_), \Phi_{(\_)}) \circ \mathcal{P}$ such that for every $a : A \to A'$ in $\boldsymbol{A}$, $\xi(\boldsymbol{A}, \Phi)(A) = (ev_A, \gamma(\boldsymbol{A}, \Phi))$ where

$$\begin{tikzcd}
\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}}) \arrow[rr, "\Phi"] \arrow[d, "\boldsymbol{Mod}(\boldsymbol{Set}^{\xi(\boldsymbol{A},\Phi)})"'] & & \boldsymbol{A} \arrow[d, "\xi(\boldsymbol{A},\Phi)"] \arrow[dl, "\gamma(\boldsymbol{A},\Phi)"'] \\
\boldsymbol{Mod}(\boldsymbol{Set}^{\mathcal{P}(\boldsymbol{A},\Phi)}) \arrow[r, "\Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}"'] & {} & \boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A}, \Phi))
\end{tikzcd}$$

is such that for every $\mathcal{M}$ in $\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}})$ and (H, φ) in $\mathcal{P}(\boldsymbol{A}, \Phi)$ we have

$$\gamma(\boldsymbol{A}, \Phi)\mathcal{M}(H, \varphi) = \varphi\mathcal{M}$$

and $\xi(\boldsymbol{A}, \Phi)(a) = ev_a$, and counit $\zeta : \mathcal{P} \circ (\boldsymbol{Mod}(\_), \Phi_{(\_)}) \to id_{\boldsymbol{PRETOP}^{op}}$ such that for every pretopos $\boldsymbol{P}$, $\zeta\boldsymbol{P} : \boldsymbol{P} \to \mathcal{P}(\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}})$ is $\zeta\boldsymbol{P}(P) = ev_P$ and $\zeta\boldsymbol{P}(p) = ev_p$ for every $p : P \to P'$ in $\boldsymbol{P}$.

This 2-adjunction induces the 2-monad $\boldsymbol{S} = (S, \xi, \nu)$ where $S : \boldsymbol{T}\text{-}ALG \to \boldsymbol{T}\text{-}ALG$ is the composition

$$\boldsymbol{T}\text{-}ALG \xrightarrow{\mathcal{P}} \boldsymbol{PRETOP}^{op} \xrightarrow{(\boldsymbol{Mod}(\_), \Phi_{(\_)})} \boldsymbol{T}\text{-}ALG$$

ξ is the unit and $\nu(\boldsymbol{A}, \Phi)(\mathcal{L})(H, \varphi) = \mathcal{L}(ev_H)$ for every

$$\mathcal{L} \ \textit{in}\ \boldsymbol{Mod}(\mathcal{P}(\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A}, \Phi)), \Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}))$$

and $(H, \varphi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})$ in $\boldsymbol{T}\text{-}ALG$.

We consider the 2-category $\boldsymbol{S}\text{-}ALG$ of strict $\boldsymbol{S}$-algebras and homomorphisms of $\boldsymbol{S}$-algebras. This category has the same description given in the previous section for $\boldsymbol{T}$ with $\boldsymbol{S}$ in place of $\boldsymbol{T}$ and $\boldsymbol{T}\text{-}ALG$ in place of $\boldsymbol{CAT}$. For later reference we explicitly describe this category. An object of $\boldsymbol{S}\text{-}ALG$ is of the form $((\boldsymbol{A}, \Phi), (\Theta, \theta))$ or simply $(\boldsymbol{A}, \Phi, (\Theta, \theta))$ where $(\boldsymbol{A}, \Phi)$ is an object of $\boldsymbol{T}\text{-}ALG$ and

$$(\Theta, \theta) : (\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A}, \Phi)), \Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}) \to (\boldsymbol{A}, \Phi)$$

makes the corresponding diagrams for an $\boldsymbol{S}$-algebra commute. If we have another $\boldsymbol{S}$-algebra $(\boldsymbol{B}, \Psi, (X, \chi))$ a morphism is $((H, \varphi), s) : (\boldsymbol{A}, \Phi, (\Theta, \theta)) \to (\boldsymbol{B}, \Psi, (X, \chi))$

where $(H, \varphi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{B}, \Psi)$ is a morphism in $\boldsymbol{T}\text{-}ALG$ and s is a natural transformation

$$\begin{tikzcd}
(\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi)), \Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}) \arrow[r, "{(\Theta,\theta)}"] \arrow[d, "{(\boldsymbol{Mod}(\mathcal{P}(H,\varphi)), =)}"'] & (\boldsymbol{A}, \Phi) \arrow[d, "{(H,\varphi)}"] \arrow[dl, phantom, "s"] \\
(\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{B},\Psi)), \Phi_{\mathcal{P}(\boldsymbol{B},\Psi)}) \arrow[r, "{(X,\chi)}"'] & (\boldsymbol{B}, \Psi)
\end{tikzcd}$$

that satisfies the usual coherence conditions. s being a 2-cell in $\boldsymbol{T}\text{-}ALG$ means that

(4.9)

$$\begin{tikzcd}
\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi))}) \arrow[rr, "\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\mathcal{P}(H,\varphi))})"] \arrow[drr, "\boldsymbol{Mod}(\boldsymbol{Set}^{\Theta})"] \arrow[dd, "\Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}"'] & & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{B},\Psi))}) \arrow[rr, "\boldsymbol{Mod}(\boldsymbol{Set}^{X})"] & & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{B}}) \arrow[dd, "\Psi"] \\
& & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}}) \arrow[urr, "\boldsymbol{Mod}(\boldsymbol{Set}^{H})"'] \arrow[d, "\Phi"] \arrow[ur, phantom, "\Uparrow \boldsymbol{Mod}(\boldsymbol{Set}^{s})"] & & \\
\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi)) \arrow[rr, "\Theta"'] \arrow[urr, phantom, "\theta"] & & \boldsymbol{A} \arrow[rr, "H"'] \arrow[uurr, phantom, "\varphi"] & & \boldsymbol{B}
\end{tikzcd}$$

equals

(4.10)

$$\begin{tikzcd}
\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi))}) \arrow[rr, "\boldsymbol{Mod}(\boldsymbol{Set}^{\mathcal{P}(H,\varphi)})"] \arrow[dd, "\Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}"'] & & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{B},\Psi))}) \arrow[rr, "\boldsymbol{Mod}(\boldsymbol{Set}^{X})"] \arrow[d, "\Phi_{\mathcal{P}(\boldsymbol{B},\Psi)}"] & & \boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{B}}) \arrow[dd, "\Psi"] \\
& & \boldsymbol{Mod}(\mathcal{P}(\boldsymbol{B},\Psi)) \arrow[drr, "X"] \arrow[urr, phantom, "\chi"] & & \\
\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi)) \arrow[urr, "\boldsymbol{Mod}(\mathcal{P}(H,\varphi))"] \arrow[drr, "\Theta"'] & & & & \boldsymbol{B}. \\
& & \boldsymbol{A} \arrow[urr, "H"'] \arrow[uu, phantom, "\Uparrow s"] & &
\end{tikzcd}$$

A 2-cell $\tau : ((H\varphi), s) \to ((K, \psi), t) : (\boldsymbol{A}, \Phi, (\Theta, \theta)) \to (\boldsymbol{B}, \Psi, (X, \chi))$ is a 2-cell

$\tau : (\boldsymbol{A}, \Phi) \to (\boldsymbol{B}, \Psi)$ in $\boldsymbol{T}$-ALG such that

$$
\begin{tikzcd}[column sep=huge]
(\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi)), \Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}) \arrow[rr, bend left, "\boldsymbol{Mod}(\mathcal{P}(K,\psi))"] \arrow[rr, bend right, "\boldsymbol{Mod}(\mathcal{P}(K,\psi))"'] \arrow[rr, phantom, "\Uparrow \boldsymbol{Mod}(\tau)"] \arrow[d, "{(\Theta,\theta)}"'] & & (\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{B},\Psi)), \Phi_{\mathcal{P}(\boldsymbol{B},\Psi)}) \arrow[d, shift left, "{(X,\chi)}"] \\
(\boldsymbol{A}, \Phi) \arrow[rr, "{(H,\varphi)}"'] & & (\boldsymbol{B}, \Psi) \arrow[u, shift left=3, "s"]
\end{tikzcd}
\tag{4.11}
$$

equals

$$
\begin{tikzcd}[column sep=huge]
(\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi)), \Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}) \arrow[rr, "\boldsymbol{Mod}(\mathcal{P}(K,\psi))"] \arrow[d, "{(\Theta,\theta)}"'] & & (\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{B},\Psi)), \Phi_{\mathcal{P}(\boldsymbol{B},\Psi)}) \arrow[d, shift left, "{(X,\chi)}"] \\
(\boldsymbol{A}, \Phi) \arrow[rr, bend left, "{(K,\psi)}"] \arrow[rr, bend right, "{(H,\varphi)}"'] \arrow[rr, phantom, "\Uparrow \tau"] & & (\boldsymbol{B}, \Psi) \arrow[u, shift left=3, "t"]
\end{tikzcd}
\tag{4.12}
$$

Next we define a functor $Z : \boldsymbol{S}\text{-}ALG \to \boldsymbol{UC}$. First consider the composition

$$
\boldsymbol{S}\text{-}ALG \xrightarrow{U} \boldsymbol{T}\text{-}ALG \xrightarrow{W} \boldsymbol{PUC}
$$

where U denotes the forgetful functor and W was defined above. Given an $\boldsymbol{S}$-algebra $(\boldsymbol{A}, \Phi, (\Theta, \theta))$ the underlying pre-ultracategory of $Z(\boldsymbol{A}, \Phi, (\Theta, \theta))$ is $W(\boldsymbol{A}, \Phi)$. Let $\underline{\boldsymbol{G}}$ be an ultragraph, k and l nodes of $\underline{\boldsymbol{G}}$ and δ an ultramorphism

$$
\begin{tikzcd}[column sep=huge]
\boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}}) \arrow[r, bend left, "ev_k"] \arrow[r, bend right, "ev_l"'] \arrow[r, phantom, "\Downarrow \delta"] & \boldsymbol{Set}
\end{tikzcd}
$$

on $\underline{\boldsymbol{Set}}$. We want to define $\delta_{Z(\boldsymbol{A},\Phi,(\Theta,\theta))}$

$$
\begin{tikzcd}[column sep=huge]
\boldsymbol{UD}(\underline{\boldsymbol{G}}, W(\boldsymbol{A}, \Phi)) \arrow[r, bend left, "ev_k"] \arrow[r, bend right, "ev_l"'] \arrow[r, phantom, "\Downarrow \delta_{Z(\boldsymbol{A},\Phi,(\Theta,\theta))}"] & \boldsymbol{A}.
\end{tikzcd}
$$

Let $D \in \boldsymbol{UD}(\underline{\boldsymbol{G}}, W(\boldsymbol{A}, \Phi))$. Define $\widehat{D} : \mathcal{P}(\boldsymbol{A}, \Phi) \to \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}})$ such that

$$\widehat{D}(H, \varphi) = H \circ D : \boldsymbol{G} \to \boldsymbol{A}$$

and $\widehat{D}(\tau) = \tau D$ for every $\tau : (H, \varphi) \to (K, \psi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})$. We have to show that $H \circ D$ is an ultradiagram. Let $\beta \in \boldsymbol{G}^b$. Since D is an ultradiagram we have an isomorphism $D(\beta) \to [\mathcal{U}_\beta]_{W(\boldsymbol{A},\Phi)}(\langle D(g_\beta(i))\rangle_{I_\beta})$ and therefore we have an isomorphism

$$H(D(\beta)) \xrightarrow{\simeq} H(\textstyle\prod_{I_\beta} D(g_\beta(i))/\mathcal{U}_\beta) \xrightarrow{\varphi[\mathcal{U}_\beta]\eta \boldsymbol{A}^I} \textstyle\prod_{I_\beta} H(D(g_\beta(i)))/\mathcal{U}_\beta.$$

Next we have to show that τD is a morphism of ultradiagrams but it follows easily from the fact that $W(\tau)$ is a pre-ultranatural transformation that the right hand side square in the diagram

$$\begin{tikzcd}
H(D(\beta)) \arrow[r] \arrow[d, "\tau D(\beta)"] & H(\prod_{I_\beta} D(g_\beta(i))/\mathcal{U}_\beta) \arrow[r, "\varphi[\mathcal{U}_\beta]\eta \boldsymbol{A}^I"] \arrow[d, "\tau(\prod_{I_\beta} D(g_\beta(i))/\mathcal{U}_\beta)"] & \prod_{I_\beta} H(D(g_\beta(i)))/\mathcal{U}_\beta \arrow[d, "\prod_{I_\beta} \tau(D(g_\beta(i)))/\mathcal{U}_\beta"] \\
K(D(\beta)) \arrow[r] & K(\prod_{I_\beta} D(g_\beta(i))/\mathcal{U}_\beta) \arrow[r, "\psi[\mathcal{U}_\beta]\eta \boldsymbol{A}^I"'] & \prod_{I_\beta} K(D(g_\beta(i)))/\mathcal{U}_\beta
\end{tikzcd}$$

commutes while the left hand side square commutes by the naturality of τ. We have now an easy lemma.

Lemma 4.18. *The functor $\widehat{D} : \mathcal{P}(\boldsymbol{A}, \Phi) \to \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}})$ is elementary.*

□

Consider the diagram

$$\begin{tikzcd}
\mathcal{P}(\boldsymbol{A}, \Phi) \arrow[r, "\widehat{D}"] & \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}}) \arrow[r, bend left, "ev_k"] \arrow[r, bend right, "ev_l"'] \arrow[r, phantom, "\downarrow\delta"] & \boldsymbol{Set}
\end{tikzcd}$$

Notice that the top composition is $ev_{D(k)}$ and the bottom one is $ev_{D(l)}$. Since the diagram

$$\begin{tikzcd} (\boldsymbol{A}, \Phi) \arrow[rr, "\xi(\boldsymbol{A}, \Phi)"] \arrow[dr, "id_{(\boldsymbol{A}, \Phi)}"'] & & (\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A}, \Phi)), \Phi_{\mathcal{P}(\boldsymbol{A}, \Phi)}) \arrow[dl, "\Theta"] \\ & (\boldsymbol{A}, \Phi) & \end{tikzcd}$$

commutes we have

$$D(k) = \Theta(ev_k \circ \widehat{D}) \xrightarrow{\Theta(\delta \widehat{D})} \Theta(ev_l \circ \widehat{D}) = D(l).$$

Define $\delta_{Z(\boldsymbol{A}, \Phi, (\Theta, \theta))}(D) = \Theta(\delta \widehat{D})$.

Lemma 4.19. $\delta_{Z(\boldsymbol{A}, \Phi, (\Theta, \theta))} : ev_k \to ev_l : \boldsymbol{UD}(\underline{\boldsymbol{G}}, W(\boldsymbol{A}, \Phi)) \to \boldsymbol{A}$ *defined above is a natural transformation.*

Proof. Let $d : D \to D' : \underline{\boldsymbol{G}} \to W(\boldsymbol{A}, \Phi)$ be a morphism of ultradiagrams. We can induce then the natural transformation $\widehat{d} : \widehat{D} \to \widehat{D'} : \mathcal{P}(\boldsymbol{A}, \Phi) \to \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}})$ such that $\widehat{d}(H, \varphi) = Hd$. Consider

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{A}, \Phi) \arrow[r, bend left, "\widehat{D}"] \arrow[r, bend right, "\widehat{D'}"'] \arrow[r, phantom, "\Downarrow \widehat{d}"] & \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}}) \arrow[r, bend left, "ev_k"] \arrow[r, bend right, "ev_l"'] \arrow[r, phantom, "\Downarrow \delta"] & \boldsymbol{Set}. \end{tikzcd}$$

This gives us a commutative square

$$\begin{tikzcd} ev_k\widehat{D} \arrow[r, "\delta\widehat{D}"] \arrow[d, "ev_k\widehat{d}"'] & ev_l\widehat{D} \arrow[d, "ev_l\widehat{d}"] \\ ev_k\widehat{D'} \arrow[r, "\delta\widehat{D'}"'] & ev_l\widehat{D'} \end{tikzcd}$$

in $\boldsymbol{Mod}(\boldsymbol{Set}^{\mathcal{P}(\boldsymbol{A}, \Phi)})$. Notice that $ev_k\widehat{d} = ev_{dk}$ and therefore $\Theta(ev_k\widehat{d}) = dk$. Similarly $\Theta(ev_l\widehat{d}) = dl$. Applying Θ to the square above we obtain

$$\begin{tikzcd} D(k) \arrow[r, "\delta_{Z(\boldsymbol{A}, \Phi, (\Theta, \theta))} D"] \arrow[d, "dk"'] & D(l) \arrow[d, "dl"] \\ D'(k) \arrow[r, "\delta_{Z(\boldsymbol{A}, \Phi, (\Theta, \theta))} D'"'] & D'(l) \end{tikzcd}$$

□

With this definition of $\delta_Z(\boldsymbol{A}, \Phi, (\Theta, \theta))$ we have that $Z(\boldsymbol{A}, \Phi, (\Theta, \theta))$ is an ultra-category

Proposition 4.20. *For every morphism* $((H, \varphi), s) : (\boldsymbol{A}, \Phi, (\Theta, \theta)) \to (\boldsymbol{B}, \Psi, (X, \chi))$ *in* $\boldsymbol{S}$-*ALG we have that the pre-ultrafunctor* $H : W(\boldsymbol{A}, \Phi) \to W(\boldsymbol{B}, \Psi)$ *is an ultra-functor* $H : Z(\boldsymbol{A}, \Phi, (\Theta, \theta)) \to Z(\boldsymbol{B}, \Psi, (X, \chi))$

Proof. Let $\delta : ev_k \to ev_l : \underline{\boldsymbol{G}} \to \underline{\boldsymbol{Set}}$ be an ultramorphism. We have to show that

$$H\delta_{Z(\boldsymbol{A},\Phi,(\Theta,\theta))} = \delta_{Z(\boldsymbol{B},\Psi,(X,\chi))}\boldsymbol{UD}(\underline{\boldsymbol{G}}, W(H, \varphi))$$

That is we want to show that $H(\Theta(\delta\widehat{D})) = X(\delta\widehat{HD})$ for every $D \in \boldsymbol{UD}(\underline{\boldsymbol{G}}, W(\boldsymbol{A}, \Phi))$. Observe first that the diagram

$$\begin{tikzcd} \mathcal{P}(\boldsymbol{B}, \Psi) \arrow[rr, "\widehat{HD}"] \arrow[dr, "\mathcal{P}(H,\varphi)"'] & & \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}}) \arrow[dl, "\widehat{D}"] \\ & \mathcal{P}(\boldsymbol{A}, \Phi) & \end{tikzcd}$$

commutes. Then $\delta\widehat{HD} = \delta\widehat{D}\mathcal{P}(H, \varphi)$. Using the naturality of s we obtain the following commutative diagram

$$\begin{tikzcd} H(\Theta(ev_k\widehat{D})) \arrow[r, "s\, ev_k\widehat{D}"] \arrow[d, "H(\Theta(\delta\widehat{D}))"'] & X(ev_k\widehat{D}\mathcal{P}(H, \varphi)) \arrow[d, "X(\delta\widehat{HD})"] \\ H(\Theta(ev_l\widehat{D})) \arrow[r, "s\, ev_l\widehat{D}"'] & X(ev_l\widehat{D}\mathcal{P}(H, \varphi)) \end{tikzcd}$$

Using the fact that s satisfies the coherence axiom involving the unit and that $ev_k\widehat{D} = ev_D(k)$ we have that $s\, ev_k\widehat{D} = id_{HD(k)}$ □

Define $Z((H, \varphi), s) = H$.

It is clear that for a 2-cell $\tau : ((H, \varphi), s) \to ((K, \psi), t)$ we have that $\tau : W(H, \varphi) \to W(K, \psi)$ is a pre-ultranatural transformation, therefore

$$\tau : Z((H, \varphi), s) \to Z((K, \psi), t)$$

is an ultrafunctor. Define $Z(\tau) = \tau$.

This completes the definition of $Z : \boldsymbol{S}\text{-}ALG \to \boldsymbol{UC}$. So we have a commutative diagram of 2-functors

$$\begin{tikzcd} \boldsymbol{S}\text{-}ALG \arrow[r, "Z"] \arrow[d] & \boldsymbol{UC} \arrow[d] \\ \boldsymbol{T}\text{-}ALG \arrow[r, "W"'] & \boldsymbol{PUC} \end{tikzcd}$$

where the vertical arrows are forgetful 2-functors.

We obtain a comparison functor $\boldsymbol{PRETOP}^{op} \to (\boldsymbol{S}\text{-}ALG)_s$ whose composition with the inclusion $(\boldsymbol{S}\text{-}ALG)_s \to \boldsymbol{S}\text{-}ALG$ we call

$$(\boldsymbol{Mod}(\ ), \Phi_{(-)}, (\Theta_{(-)}, =)) : \boldsymbol{PRETOP}^{op} \to \boldsymbol{S}\text{-}ALG.$$

It is easy to see that for every pretopos $\boldsymbol{P}$, every model $\mathcal{M}$ in $\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}}))$ and every P in $\boldsymbol{P}$ we have that $\Theta_{\boldsymbol{P}}(\mathcal{M})(P) = \mathcal{M}(ev_P)$

Proposition 4.21. *The functor $Z : \boldsymbol{S}\text{-}ALG \to \boldsymbol{UC}$ is such that for every pretopos $\boldsymbol{P}$ we have $Z(\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}}, (\Theta_{\boldsymbol{P}}, =)) = \underline{\underline{\boldsymbol{Mod}}}(\boldsymbol{P})$*

Proof. By Proposition 4.16 we already know that the underlying category of $Z(\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}}, (\Theta_{\boldsymbol{P}}, =))$ is $\underline{\boldsymbol{Mod}}(\boldsymbol{P})$. So all we have to check is the ultramorphisms. Let $\delta : ev_k \to ev_l : \boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Set}}) \to \boldsymbol{Set}$ be an ultamorphism and let D be an ultradiagram in $\boldsymbol{UD}(\underline{\boldsymbol{G}}, \underline{\boldsymbol{Mod}}(\boldsymbol{P}))$. Then for every P in $\boldsymbol{P}$ we have

$$\delta_{Z(\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}}, (\Theta_{\boldsymbol{P}}, =))} D(P) = \Theta_{\boldsymbol{P}}(\delta\widehat{D})(P) = \delta\widehat{D}(ev_P) = \delta(ev_P \circ D) = \delta D(\_)(P).$$

□

As before, when $\boldsymbol{P}$ is the full subcategory of $\boldsymbol{Set}^{\boldsymbol{Set}_0}$ consisting of the finitely generated functors we have that $(\boldsymbol{Mod}(\boldsymbol{P}), \Phi_{\boldsymbol{P}}, (\Theta_{\boldsymbol{P}}, =))$ is essentially

$$(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}}, (X_{\boldsymbol{Set}}, =))$$

where $X_{\boldsymbol{Set}} = ev_{id_{\boldsymbol{Set}}}$. As a consequence of the above proposition we have

$$Z(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}}, (X_{\boldsymbol{Set}}, =)) = \underline{\underline{\boldsymbol{Set}}}.$$

Proposition 4.22. *For every object* $(\boldsymbol{A}, \Phi, (\Theta, \theta))$ *the category*

$$\boldsymbol{S}\text{-}ALG((\boldsymbol{A}, \Phi, (\Theta, \theta)), (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}}, (X_{\boldsymbol{Set}}, =)))$$

is a pretopos and for every morphism

$$(\boldsymbol{A}, \Phi, (\Theta, \theta)) \xrightarrow{((H, \varphi), s)} (\boldsymbol{B}, \Psi, (X, \chi))$$

in $\boldsymbol{S}$-ALG *the functor* $\boldsymbol{S}\text{-}ALG(((H, \varphi), s), (\boldsymbol{Set}, \Phi_{\boldsymbol{Set}}, (X_{\boldsymbol{Set}}, =)))$ *is an elementary functor. Furthermore the corresponding limits and colimits are calculated pointwise.*

Proof. We do binary coproducts to illustrate the point, all the other constructions are similar. Suppose we have

$$(H, \varphi, s), (K, \psi, t) : (\boldsymbol{A}, \Phi, (\Theta, \theta)) \to (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}}, (X_{\boldsymbol{Set}}, =))$$

in $\boldsymbol{S}\text{-}ALG((\boldsymbol{A}, \Phi, (\Theta, \theta)), (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}}, (X_{\boldsymbol{Set}}, =)))$. Consider first the coproduct

$$(H, \varphi) \coprod (K, \psi) = (H \coprod K, \varphi')$$

in $\boldsymbol{T}\text{-}ALG((\boldsymbol{A}, \Phi), (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}}))$ where $\varphi' M$ is the composition

$$(H \coprod K)(\Phi M) = H \Phi M \coprod K \Phi M \xrightarrow{\varphi M \coprod \psi M} M H \coprod M K \xrightarrow{\simeq} M(H \coprod K)$$

for every M in $\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{A}})$. We want to define s' in

$$\begin{tikzcd}
(\boldsymbol{Mod}((\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(\boldsymbol{A}, \Phi)}), \Phi_{\mathcal{P}(\boldsymbol{A}, \Phi)}) \arrow[r, "(\Theta, \theta)"] \arrow[d, "\boldsymbol{Mod}((\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(H \coprod K, \varphi')})"'] & (\boldsymbol{A}, \Phi) \arrow[d, "X_{\boldsymbol{Set}}"] \\
(\boldsymbol{Mod}((\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})}), \Phi_{\mathcal{P}(\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})}) \arrow[r, "(H \coprod K, \varphi')"'] & (\boldsymbol{Set}, \Psi_{\boldsymbol{Set}}) \arrow[u, "s'"]
\end{tikzcd}$$

Given $\mathcal{M}$ in $\boldsymbol{Mod}((\boldsymbol{Set}, \Psi_{\boldsymbol{Set}})^{(\boldsymbol{A}, \Phi)})$ define $s' \mathcal{M}$ as the composition

$$\begin{tikzcd}
H \Theta \mathcal{M} \coprod K \Theta \mathcal{M} \arrow[r, "\langle s\mathcal{M}, t\mathcal{M} \rangle"] \arrow[d, equal] & \mathcal{M}(H, \varphi) \coprod \mathcal{M}(K, \psi) \arrow[r, "\simeq"] & \mathcal{M}((H, \varphi) \coprod (K, \psi)) \arrow[d, equal] \\
(H \coprod K) \Theta \mathcal{M} & & \mathcal{M}(H \coprod K, \varphi')
\end{tikzcd}$$

It is easy to see that s' is natural. We show now that the composition corresponding to diagram 4.9 and the composition corresponding to diagram 4.10 are equal. Let $\mathcal{L}$ in $\boldsymbol{Mod}(\boldsymbol{Set}^{\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi))})$, then from 4.9 and 4.10 for s and t we have that

$$\mathcal{L}(s) \circ \varphi(\mathcal{L} \circ \boldsymbol{Set}^{\Theta}) \circ H\theta(\mathcal{L}) = s\Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}(\mathcal{L})$$

$$\mathcal{L}(t) \circ \psi(\mathcal{L}\boldsymbol{Set}^{\Theta}) \circ K\theta(\mathcal{L}) = t\Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}(\mathcal{L}).$$

With these two equations it is not hard to see that

$$\mathcal{L}(s') \circ \varphi'(\mathcal{L} \circ \boldsymbol{Set}^{\Theta}) \circ H\textstyle\coprod K\theta(\mathcal{L}) = s'\Phi_{\mathcal{P}(\boldsymbol{A},\Phi)}(\mathcal{L})$$

Therefore s' is a 2-cell in $\boldsymbol{T}$-ALG. We have a coproduct diagram

$$(H,\varphi) \xrightarrow{i_H} (H\textstyle\coprod K, \varphi') \xleftarrow{i_K} (K,\psi)$$

in $\boldsymbol{T}$-ALG. To show that $i_H : (H,\varphi,s) \to (H\coprod K, \varphi', s')$ is a 2-cell in $\boldsymbol{S}$-ALG all we have to show (according to 4.11 and 4.12) is that

$$H\Theta\mathcal{M} \xrightarrow{s\mathcal{M}} \mathcal{M}(H,\varphi) \xrightarrow{\mathcal{M}(i_H)} \mathcal{M}(H\textstyle\coprod K, \varphi')$$

equals

$$\begin{tikzcd}
H\Theta\mathcal{M} \arrow[r, "i_H\Theta"] & (H\coprod K)\Theta\mathcal{M} = H\Theta\mathcal{M}\coprod K\Theta\mathcal{M} \arrow[r, "s\mathcal{M}\coprod t\mathcal{M}"] & \mathcal{M}(H,\varphi)\coprod\mathcal{M}(K,\psi) \arrow[d, "\simeq"] \\
 & & \mathcal{M}(H\coprod K, \varphi')
\end{tikzcd}$$

for every $\mathcal{M}$ in $\boldsymbol{Mod}(\mathcal{P}(\boldsymbol{A},\Phi))$, but this is readily seen to be the case. The universal property also follows easily. $\square$

Chapter 5

Algebras Over Łos Categories

In 4.5.3 we saw how to obtain pre-ultracategories from algebras over $\boldsymbol{CAT}$, that is, we constructed pre-ultrafunctors with the help of the structure map. We saw as well how to obtain some of the ultramorphisms. We needed however a second monad to be able to introduce general ultramorphisms. In this chapter we avoid the first monad by working in the category $\mathfrak{Los}$. Notice that we introduced this category with the express purpose of dealing with ultraproducts. With the category $\mathfrak{Los}$ we also obtain some of the ultramorphisms, however we do not see how to get the general ultramorphisms. In this short chapter we define a monad over $\mathfrak{Los}$ and show how we can obtain the general ultramorphisms for algebras over this monad. On the one hand this simplifies the notation since we are dealing only with one monad and the rest of the structure is given by the $\boldsymbol{Top}$-indexing, on the other it provides a nice setting in which, we hope, the other side of Makkai's duality can be proven, namely characterize those categories that are of the form $\boldsymbol{Mod}(\boldsymbol{P})$ for a small pretopos $\boldsymbol{P}$.

Notation Given a $\boldsymbol{Top}$-indexed functor $F : \mathcal{A} \to \mathcal{B}$ and a discrete topological space I we denote by $F^I : \mathcal{A}^I \to \mathcal{B}^I$ the corresponding F for the topological space I as opposed to the product functor $\prod_I \mathcal{A}^1 \xrightarrow{\prod_I F^1} \prod_I \mathcal{B}^1$ that we denote by $F^I : \boldsymbol{A}^I \to \boldsymbol{B}^I$.

5.1 Łos Categories and Pre-Ultracategories

We define first a functor $\mathfrak{Los} \to \boldsymbol{PUC}$. Given a category $\mathcal{A}$ in $\mathfrak{Los}$ we construct a pre-ultracategory as follows. The underlying category is $\boldsymbol{A} = \mathcal{A}^1$. Given an ultrafilter

$(I,\mathcal{U})$ denote by $f : I \to I_{\mathcal{U}}$ the embedding and define $[\mathcal{U}]_{\boldsymbol{A}}$ as the composition

$$\boldsymbol{A}^I \xrightarrow{\simeq} \mathcal{A}^I \xrightarrow{f_*} \mathcal{A}^{I_{\mathcal{U}}} \xrightarrow{\mathcal{U}^*} \boldsymbol{A}$$

where the first arrow is given by the fact that $\mathcal{A}$ is in $\boldsymbol{Top}$-IND (definition 3.11). If $F : \mathcal{A} \to \mathcal{B}$ in $\mathfrak{Los}$ consider $F^1 = F : \boldsymbol{A} \to \boldsymbol{B}$ and define the natural isomorphism $[\mathcal{U}, F]$ as the pasting

$$\begin{tikzcd}
\boldsymbol{A}^I \arrow[r, "\simeq"] \arrow[d, "F^I"'] & \mathcal{A}^I \arrow[r, "f_*"] \arrow[d, "F^I"] \arrow[dl, phantom, "\simeq"] & \mathcal{A}^{I_{\mathcal{U}}} \arrow[r, "\mathcal{U}^*"] \arrow[d, "F^{I_{\mathcal{U}}}"] \arrow[dl, phantom, "\simeq"] & \boldsymbol{A} \arrow[d, "F"] \arrow[dl, phantom, "\simeq"] \\
\boldsymbol{B}^I \arrow[r, "\simeq"'] & \mathcal{B}^I \arrow[r, "f_*"'] & \mathcal{B}^{I_{\mathcal{U}}} \arrow[r, "\mathcal{U}^*"'] & \boldsymbol{B}
\end{tikzcd}$$

where the two natural isomorphisms on the left are given by the fact that F is in $\mathfrak{Los}$ (definitions 3.11 and 3.14) and the one on the right is given by F being $\boldsymbol{Top}$-indexed. It is easy to see that this construction does define a functor $\mathfrak{Los} \to \boldsymbol{PUC}$.

If $\boldsymbol{P}$ is a pretopos then it is clear that the pre-ultractegory we obtain as the image of $\mathcal{MOD}(\boldsymbol{P})$ under this functor is $\underline{\boldsymbol{Mod}}(\boldsymbol{P})$, as a particular case we have that the image of $\mathcal{SET}$ is $\underline{\boldsymbol{Set}}$.

5.2 Algebras Over Łos Categories

From Proposition 3.13 and the remark after the proof we have a 2-functor

$$\mathfrak{Los}(\_, \mathcal{SET}) : \mathfrak{Los} \to \boldsymbol{PRETOP}^{op}.$$

On the other hand we have the 2-functor

$$\mathcal{MOD}(\_) : \boldsymbol{PRETOP}^{op} \to \mathfrak{Los}.$$

We obtain a 2-adjunction

$$\begin{tikzcd}
\boldsymbol{PRETOP}^{op} \arrow[d, shift left=2, "\mathcal{MOD}(_)"] \\
\mathfrak{Los} \arrow[u, shift left=2, "\mathfrak{Los}(_, \mathcal{SET})"]
\end{tikzcd}$$

whose counit $\varepsilon\boldsymbol{P} : \boldsymbol{P} \to \mathfrak{Los}(\mathcal{MOD}(\boldsymbol{P}), \mathcal{SET})$ is $P \mapsto ev_P$ for any pretopos $\boldsymbol{P}$ and P in $\boldsymbol{P}$. The unit $\eta\mathcal{A} : \mathcal{A} \to \mathcal{MOD}(\mathfrak{Los}(\mathcal{A}, \mathcal{SET}))$ is such that for any $\mathcal{A} \in \mathfrak{Los}$ any topological space X, any A in $\mathcal{A}^X$ and any $\tau : F \to G$ in $\mathfrak{Los}(\mathcal{A}, \mathcal{SET})$ we have $(\eta\mathcal{A})^X(A)(F) = F^X(A)$ and $(\eta\mathcal{A})^X(A)(\tau) = \tau^X(A)$. It is easy to see that for every A in $\mathcal{A}^X$ the functor $\eta\mathcal{A}^X(A) : \mathfrak{Los}(\mathcal{A}, \mathcal{SET}) \to Sh(X)$ is elementary. We have to show that for every $\mathcal{A}$ in $\mathfrak{Los}$ the functor $\eta\mathcal{A}$ is indeed in $\mathfrak{Los}$. We show first that it is $\boldsymbol{Top}$-indexed. Given a continuous function $f : Y \to X$ we need a transition isomorphism $\eta\mathcal{A}^Y \circ f^* \to f^* \circ \eta\mathcal{A}^X$. Let A in $\mathcal{A}^X$ and F in $\mathfrak{Los}(\mathcal{A}, \mathcal{SET})$ then we want an isomorphism $f^*\eta\mathcal{A}^X(A)(F) \to \eta\mathcal{A}^Y(f^*A)(F)$. That is $f^*F^XA \to F^Yf^*A$. Since F is $\boldsymbol{Top}$-indexed we have an isomorphism $f^*F^XA \to F^Yf^*A$ that we can use to define the isomorphism we are looking for.

It is easy to see that $\eta\mathcal{A}$ is Los. Assume $f : Y \to X$ is ultrafinite in $\boldsymbol{Top}$, we need to show that

$$\eta\mathcal{A}^X f_* \xrightarrow{unit\,\eta\mathcal{A}^X f_*} f_*f^*\eta\mathring{\mathcal{A}}^X f_* \xrightarrow{\simeq} f_*\eta\mathcal{A}^Y f^* f_* \xrightarrow{f_*\eta\mathcal{A}^Y\, counit} f_*\eta\mathcal{A}^Y$$

is an isomorphism. Take A in $\mathcal{A}^Y$ and F in $\mathfrak{Los}(\mathcal{A}, \mathcal{SET})$ and if we apply the above composition at A at F we obtain

$$F^X f_*A \xrightarrow{unit\, F^X f_*A} f_*f^*F^X f_*A \xrightarrow{\simeq} f_*F^Y f^* f_*A \xrightarrow{f_*F^Y\, counit\, A} f_*\eta F^Y A$$

that is an isomorphism since F is Los.

We obtain therefore a 2-monad $\boldsymbol{T} = (T, \eta, \mu)$ over $\mathfrak{Los}$. Consider the category $\boldsymbol{T}$-ALG of $\boldsymbol{T}$-algebras. We define now a 2-functor $\boldsymbol{T}$-$ALG \to \boldsymbol{UC}$. Let $(\mathcal{A}, \Phi)$ be a $\boldsymbol{T}$-algebra, consider first the pre-ultracategory $\underline{\boldsymbol{A}}$ constructed from $\mathcal{A}$ as in 5.1. Notice that for any ultragraph $\boldsymbol{G}$ composing with $\eta\mathcal{A}^1 : \boldsymbol{A} \to \boldsymbol{Mod}(\mathfrak{Los}(\mathcal{A}, \mathcal{SET}))$ induces a functor $\boldsymbol{UD}(\boldsymbol{G}, \boldsymbol{A}) \to \boldsymbol{UD}(\boldsymbol{G}, \boldsymbol{Mod}(\mathfrak{Los}(\mathcal{A}, \mathcal{SET})))$. If we have an ultramorphism

$$\begin{tikzcd} \boldsymbol{UD}(\boldsymbol{G}, \boldsymbol{Set}) \arrow[r, bend left, "ev_k"] \arrow[r, bend right, "ev_\ell"'] \arrow[r, phantom, "\downarrow\delta"] & \boldsymbol{Set} \end{tikzcd}$$

over $\boldsymbol{Set}$ define $\delta_{\boldsymbol{A}} = \Phi^1 \circ \delta_{\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{A}, \mathcal{SET}))} \circ \boldsymbol{UD}(\boldsymbol{G}, \eta\mathcal{A}^1)$

Lemma 5.1. *If $(F, \varphi) : (\boldsymbol{A}, \Phi) \to (\boldsymbol{B}, \Psi)$ is a 1-cell in $\boldsymbol{T}$-ALG then $F : \boldsymbol{A} \to \boldsymbol{B}$ is an ultrafunctor.*

Proof. Simply put the following diagrams together

$$\begin{tikzcd}
UD(G,\boldsymbol{A}) \arrow[r, "UD(G{,}\eta\mathcal{A}^1)"] \arrow[d, "UD(G{,}F)"'] & UD(G,\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{A},\mathcal{SET}))) \arrow[d, "UD(G{,}\boldsymbol{Mod}(\mathfrak{Los}(F{,}\mathcal{SET})))"] \\
UD(G,\boldsymbol{B}) \arrow[r, "UD(G{,}\eta\mathcal{B}^1)"'] & UD(G,\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{B},\mathcal{SET})))
\end{tikzcd}$$

$$\begin{tikzcd}
UD(G,\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{A},\mathcal{SET}))) \arrow[r, bend left, "ev_k"] \arrow[r, bend right, "ev_\ell"'] \arrow[r, phantom, "\downarrow\delta_{\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{A},\mathcal{SET}))}"] \arrow[d, "UD(G{,}\boldsymbol{Mod}(\mathfrak{Los}(F{,}\mathcal{SET})))"] & \boldsymbol{Mod}(\mathfrak{Los}(\mathcal{A},\mathcal{SET})) \arrow[d, "\boldsymbol{Mod}(\mathfrak{Los}(F{,}\mathcal{SET}))"] \\
UD(G,\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{B},\mathcal{SET}))) \arrow[r, bend left, "ev_k"] \arrow[r, bend right, "ev_\ell"'] \arrow[r, phantom, "\downarrow\delta_{\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{B},\mathcal{SET}))}"] & \boldsymbol{Mod}(\mathfrak{Los}(\mathcal{B},\mathcal{SET}))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{A},\mathcal{SET})) \arrow[r, "\Phi^1"] \arrow[d, "\boldsymbol{Mod}(\mathfrak{Los}(F{,}\mathcal{SET}))"'] & \boldsymbol{A} \arrow[d, "H^1"] \arrow[dl, "\varphi^1"'] \\
\boldsymbol{Mod}(\mathfrak{Los}(\mathcal{B},\mathcal{SET})) \arrow[r, "\Psi^1"'] & \boldsymbol{B}
\end{tikzcd}$$

□

Lemma 5.2. *If $\tau : (F,\varphi) \to (G,\psi) : (\boldsymbol{A},\Phi) \to (\boldsymbol{B},\Psi)$ is a 2-cell in $\boldsymbol{T}$-ALG then $\tau : F \to G : \mathcal{A} \to \mathcal{B}$ is an ultranatural transformation.* □

We obtain a functor $\boldsymbol{T}\text{-}ALG \to \boldsymbol{UC}$. Notice that we obtain the following commutative diagram

$$\begin{tikzcd}
\boldsymbol{T}\text{-}ALG \arrow[r] \arrow[d] & \boldsymbol{UC} \arrow[d] \\
\mathfrak{Los} \arrow[r] & \boldsymbol{PUC}
\end{tikzcd}$$

where the vertical arrows are forgetful functors.

Chapter 6

Indexed Categories of Coalgebras

In this chapter we generalize a result from [11] namely that there is an equivalence between $\boldsymbol{Top}$-$ind(\mathcal{SET}, \mathcal{SET})$ and $\boldsymbol{Filt}(\boldsymbol{Set}, \boldsymbol{Set})$ given by

$$F \mapsto F^1$$

where $\boldsymbol{Filt}(\boldsymbol{Set}, \boldsymbol{Set})$ denotes the category of functors that preserve filtered colimits, and use this generalization to show that if $F : \mathcal{MOD}(\boldsymbol{P}) \to \mathcal{SET}$ is a $\boldsymbol{Top}$-indexed functor then $F^1 : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Set}$ preserves filtered colimits.

We consider a special kind of $\boldsymbol{Top}$-indexed categories, namely those that can be defined at every X as a category of coalgebras of a cotriple on the category $\boldsymbol{A}^{|X|}$ for some fixed category $\boldsymbol{A}$ (see below). The $\boldsymbol{Top}$-indexed category $\mathcal{SET}$ defined in chapter 1 is an instance of these $\boldsymbol{Top}$-indexed categories we will consider now. In particular, for every topological space X, $Sh(X)$ is equivalent to a category of coalgebras for a cotriple defined over $\boldsymbol{Set}^{|X|}$. To be able to define these categories we need products and filtered colimits in $\boldsymbol{A}$. We start with the definition of the cotriples we need.

6.1 The Cotriple G^X

Definition 6.1. Let X be a topological space, $\boldsymbol{A}$ be a category with products and filtered colimits. We define the cotriple $\boldsymbol{G}^X = (G^X, \varepsilon^X, \delta^X)$ over $\boldsymbol{A}^{|X|}$ as follows:

Define $G^X : \boldsymbol{A}^{|X|} \to \boldsymbol{A}^{|X|}$ such that $\langle A_x \rangle_{x \in X} \mapsto \langle \varinjlim_{U \ni x} \prod_{y \in U} A_y \rangle_{x \in X}$ and $\langle f_x \rangle \mapsto \langle \varinjlim_{U \ni x} \prod_{y \in U} f_y \rangle$.

Define $\epsilon^X : G^X \to 1$ such that $(\epsilon^X\langle A_x\rangle)_x$ is the unique map that makes

$$\begin{tikzcd}
\underset{U \ni x}{\varinjlim} \prod_{y \in U} A_y \arrow[rr, "(\epsilon^X\langle A_x\rangle)_x"] & & A_x \\
& \prod_{y \in U} A_y \arrow[ul, "i_U"] \arrow[ur, "\pi_x"'] &
\end{tikzcd}$$

commute.

Define $\delta^X : G^X \to G^X G^X$ such that $(\delta^X\langle A_x\rangle)_x$ is the unique map that makes

$$\begin{tikzcd}
\prod_{y \in U} A_y \arrow[r] \arrow[d] & \prod_{y \in U} \underset{V \ni y}{\varinjlim} \prod_{z \in V} A_z \arrow[d] \\
\underset{U \ni x}{\varinjlim} \prod_{y \in U} A_y \arrow[r, "(\delta^X\langle A_x\rangle)_x"'] & \underset{U \ni x}{\varinjlim} \prod_{y \in U} \underset{V \ni y}{\varinjlim} \prod_{z \in V} A_z
\end{tikzcd}$$

commute, where the top arrow is the unique arrow that makes

$$\begin{tikzcd}
\prod_{y \in U} A_y \arrow[rr] \arrow[dr, "i_U"'] & & \prod_{y \in U} \underset{V \ni y}{\varinjlim} \prod_{z \in V} A_z \arrow[dl, "\pi_y"] \\
& \underset{V \ni y}{\varinjlim} \prod_{z \in V} A_z &
\end{tikzcd}$$

commute.

It is easy to see $\boldsymbol{G^X}$ is indeed a cotriple.

6.2 Indexed Categories of Coalgebras

Now we are ready to define a ***Top***-indexed category.

Definition 6.2. Given a category $\boldsymbol{A}$ with products and filtered colimits define the ***Top***-indexed category $\mathcal{A}$ as follows:

For every topological space X, $\mathcal{A}^X$ is the category of coalgebras for the cotriple $\boldsymbol{G^X}$.

For every continuous function $f : X \to Z$ and every coalgebra

$$\langle A_z \xrightarrow{\tau_z} \underset{W \ni z}{\varinjlim} \prod_{w \in W} A_w \rangle$$

in $\mathcal{A}^Z$ define

$$f^*(\langle A_z \xrightarrow{\tau_z} \varinjlim_{W \ni z} \prod_{w \in W} A_w \rangle) = \langle A_{f(x)} \xrightarrow{\tau_{f(x)}} \varinjlim_{W \ni fx} \prod_{w \in W} A_w \to \varinjlim_{U \ni x} \prod_{y \in U} A_{fy} \rangle$$

where the last arrow above makes the diagram

$$\begin{tikzcd}
& \prod_{w \in W} A_w \arrow[r, "i_W"] \arrow[dl, "\pi_y"'] \arrow[dd] & \varinjlim_{W \ni fx} \prod_{w \in W} A_w \arrow[dd] \\
A_{fy} & & \\
& \prod_{y \in f^{-1}W} A_{fy} \arrow[r, "i_{f^{-1}W}"'] \arrow[ul, "\pi_{fy}"] & \varinjlim_{U \ni x} \prod_{y \in U} A_{fy}
\end{tikzcd}$$

commute. We call $\mathcal{A}$ the ***Top***-indexed category of coalgebras over $\boldsymbol{A}$.

It is easy to see that we have defined a ***Top***-indexed category. Furthermore, all the coherence axioms on the definition of an indexed category turn out to be equalities in this case. That is $\mathcal{A}$ is a strict ***Top***-indexed category.

We will be intrested in the case where $\boldsymbol{A} = \boldsymbol{Set}^{\boldsymbol{P}}$ for a pretopos $\boldsymbol{P}$, in this case we denote $\mathcal{A}$ by $\mathcal{SET}^{\boldsymbol{P}}$. Notice that when $\boldsymbol{P}$=**1**, we obtain the ***Top***-indexed category $\mathcal{SET}$.

6.3 Filtered Colimits and Absolute Equalizers

It is shown in [11] that the category ***Top***-*ind*($\mathcal{SET}$, $\mathcal{SET}$) of ***Top***-indexed functors from $\mathcal{SET}$ to itself is equivalent to the category ***Filt***(***Set***, ***Set***) of filtered colimit preserving functors from ***Set*** to ***Set***. It is our intention to generalize this result to the category ***Top***-*ind*($\mathcal{A}$, $\mathcal{B}$) where $\mathcal{A}$ and $\mathcal{B}$ are the ***Top***-indexed categories of coalgebras over $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively. However, to be able to do this we need more structure on the categories $\boldsymbol{A}$ and $\boldsymbol{B}$. See proposition 6.9.

Take a category $\boldsymbol{A}$ with products and filtered colimits. If $\boldsymbol{D}$ is a small directed poset, and $H : \boldsymbol{D} \to \boldsymbol{A}^{\rightrightarrows}$ is a diagram, denote Hd by

$$H_0 d \underset{h_1 d}{\overset{h_0 d}{\rightrightarrows}} H_1 d$$

for $d \in \boldsymbol{D}$. Using ideas from [12] we have that one of the properties we need is the following:

Definition 6.3. Let $\boldsymbol{A}$ be a category with products and filtered colimits, we say that filtered colimits commute with pointwise absolute equalizers if for every small directed poset $\boldsymbol{D}$ and, every diagram $H : \boldsymbol{D} \to \boldsymbol{A}^{\rightrightarrows}$ such that for every $d \in \boldsymbol{D}$, Hd has an absolute equalizer $e_d : E_d \to H_0 d$ in $\boldsymbol{A}$, and the pair

$$\varinjlim_d H_0 d \underset{\varinjlim_d h_1 d}{\overset{\varinjlim_d h_0 d}{\rightrightarrows}} \varinjlim_d H_1 d$$

also has an absolute equalizer in $\boldsymbol{A}$, we have that the diagram

$$\varinjlim_d E_d \xrightarrow{\varinjlim_d e_d} \varinjlim_d H_0 d \underset{\varinjlim_d h_1 d}{\overset{\varinjlim_d h_0 d}{\rightrightarrows}} H_1 d$$

is an equalizer diagram in $\boldsymbol{A}$.

6.4 Some Topological Spaces and Their Associated Coalgebras

Here are some definitions of topological spaces and continuous functions that we are going to need later.

Recall from section 3.5 the construction of $X_{\boldsymbol{D}}$ for any small directed poset $\boldsymbol{D}$. Consider the topological space $X_{\boldsymbol{D}}^{+}$ obtained form $X_{\boldsymbol{D}}$ by adding a point ∞ not in $X_{\boldsymbol{D}}$ and whose opens are the empty set and sets of the form $U \cup \{\infty\}$ with U a nonempty open of $X_{\boldsymbol{D}}$. The inclusion $h : X_{\boldsymbol{D}} \to X_{\boldsymbol{D}}^{+}$ is clearly continuous.

Let $(I, \mathcal{F})$ be a filter. Define the topological space $I_{\mathcal{F}}$ whose set of points is $I \cup \{a_{\mathcal{F}}\}$, with $a_{\mathcal{F}} \notin I$ and the topology given by $U \subset I \cup \{a_{\mathcal{F}}\}$ open iff $a_{\mathcal{F}} \in U$ implies that $U - \{a_{\mathcal{F}}\} \in \mathcal{F}$.

In the case when $(I, \mathcal{F})$ and $(I, \mathcal{E})$ are filters with $\mathcal{E} \subset \mathcal{F}$ we have a continuous function $h_{\mathcal{F}\mathcal{E}} : I_{\mathcal{F}} \to I_{\mathcal{E}}$ such that h restricted to I is the identity and $h_{\mathcal{F}\mathcal{E}}(a_{\mathcal{F}}) = a_{\mathcal{E}}$.

If $J \subset I$ we denote by $\mathcal{S}(J)$ the principal filter generated by J. That is, $\mathcal{S}(J) = \{K \subset I | K \supset J\}$

We will denote Sierpinski's space by S, that is, $S = \{0,1\}$ and the only nontrivial open of S is $\{1\}$.

If $j \in J \subset I$ define $h_{jJ} : S \to I_{\mathcal{S}(J)}$ such that $h_{jJ}(1) = j$ and $h_{jJ}(0) = a_{\mathcal{S}(J)}$

Consider the ***Top***-indexed category $\mathcal{A}$ defined as above. Let's take a look at the category $\mathcal{A}^X$ for X the spaces we just defined, and at the transition functors induced by the continuous functions also defined above.

First of all, if we take the topological space 1, we have that $\mathcal{A}^1$ is essentially $\boldsymbol{A}$. When we have a ***Top***-indexed functor $F : \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is defined over a category $\boldsymbol{B}$ as above, we have $F^1 : \boldsymbol{A} \to \boldsymbol{B}$. Sometimes we write F instead of F^1 when it does not lead to confusion.

It is not hard to see that $\mathcal{A}^{X_D}$ is equivalent to $\boldsymbol{A}^{\boldsymbol{D}}$.

It is clear that $\mathcal{A}^S$ is isomorphic to $\boldsymbol{A}^{\rightarrow}$.

$\mathcal{A}^{I_\mathcal{F}}$ is equivalent to the category whose objects are maps $A_{a_\mathcal{F}} \xrightarrow{\tau} \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} A_j$, where $A_{a_\mathcal{F}}$ and the A_j are objects of $\boldsymbol{A}$, and whose morphisms

$$f : (A_{a_\mathcal{F}} \xrightarrow{\tau} \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} A_j) \to (B_{a_\mathcal{F}} \xrightarrow{\rho} \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} B_j)$$

are families of morphisms $(f_{a_\mathcal{F}} : A_{a_\mathcal{F}} \to B_{a_\mathcal{F}}, \{f_j : A_j \to B_j\}_J)$ such that the diagram

$$\begin{tikzcd}
A_{a_\mathcal{F}} \arrow[r, "\tau"] \arrow[d, "f_{a_\mathcal{F}}"'] & \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} A_j \arrow[d, "\varinjlim_{J \in \mathcal{F}} \prod_{j \in J} f_j"] \\
B_{a_\mathcal{F}} \arrow[r, "\rho"'] & \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} B_j
\end{tikzcd}$$

commutes. We will use this description of $\mathcal{A}^{I_\mathcal{F}}$ systematically. In the case where $\mathcal{F} = \mathcal{S}(J_0)$ for some $J_0 \subset I$ we have that $\varinjlim_{J \in \mathcal{S}(J_0)} \prod_{j \in J} A_j = \prod_{j \in J_0} A_j$. Then an object of $\mathcal{A}^{I_{\mathcal{S}(J_0)}}$ with the description given above is a pair $(A_{\mathcal{S}(J_0)} \to \prod_{j \in J_0} A_j, \langle A_i \rangle_I)$.

Now, consider the continuous function $h_{\mathcal{F}\mathcal{E}} : I_\mathcal{F} \to I_\mathcal{E}$ defined above, we have that $h^*_{\mathcal{F}\mathcal{E}} : \mathcal{A}^{I_\mathcal{E}} \to \mathcal{A}^{I_\mathcal{F}}$ is such that $(A_{a_\mathcal{E}} \xrightarrow{\tau} \varinjlim_{J \in \mathcal{E}} \prod_{j \in J} A_j) \mapsto (A_{a_\mathcal{E}} \xrightarrow{\tau} \varinjlim_{J \in \mathcal{E}} \prod_{j \in J} A_j \to$

$\varinjlim_{J \in \mathcal{F}} \prod_{j \in J} A_j)$ where the last arrow makes the diagram

$$\begin{tikzcd}
\prod_J A_j \arrow[rr, "i_J"] \arrow[dr, "i_J"'] & & \varinjlim_{J \in \mathcal{E}} \prod_{j \in J} A_j \arrow[dl] \\
& \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} A_j &
\end{tikzcd}$$

commute for every $J \in \mathcal{E}$.

For $h_{jJ_0} : S \to I_{S(J_0)}$ we have that $h^*_{jJ_0}(A_{a_{S(J_0)}} \to \prod_{j \in J_0} A_j, \langle A_i \rangle_I) = (A_{a_{S(J_0)}} \to \prod_{j \in J_0} A_j \xrightarrow{\pi_j} A_j$

6.5 The Category $\mathcal{A}^{X^+_{\mathcal{D}}}$

When we have the topological space $X^+_{\boldsymbol{D}}$ with $\boldsymbol{D}$ a small directed poset the situation is a little bit less trivial. It is here that we use the property that filtered colimits commute with pointwise absolute equalizers. Define $L : \boldsymbol{A}^{\boldsymbol{D}} \to \boldsymbol{A}^{|X^+_{\boldsymbol{D}}|}$ such that $L(\{A_d \xrightarrow{\sigma_{dd'}} A_{d'}\}_{d \to d'}) = (\varinjlim_d A_d, \langle A_d \rangle_d)$, and if $\{f_d\} : \{A_d \xrightarrow{\sigma_{dd'}} A_{d'}\}_{d \to d'} \to \{B_d \xrightarrow{\sigma_{dd'}} B_{d'}\}_{d \to d'}$ then $L(\{f_d\}) = (\varinjlim_d f_d, \langle f_d \rangle)$

Lemma 6.1. *If $\boldsymbol{A}$ is a category with products and filtered colimits such that filtered colimits commute with pointwise absolute equalizers, $\boldsymbol{D}$ a small directed poset then the functor $L : \boldsymbol{A}^{\boldsymbol{D}} \to \boldsymbol{A}^{|X^+_{\boldsymbol{D}}|}$ defined above is cotripleable.*

Proof. We use Beck's tripleability theorem (see [13] for example). First, we need a right adjoint. Define $R : \boldsymbol{A}^{|X^+_{\boldsymbol{D}}|} \to \boldsymbol{A}^{\boldsymbol{D}}$ such that $R((A_\infty, \langle A_d \rangle)) = \{A_\infty \times \prod_{d'' \leftarrow d} A_{d''} \xrightarrow{p_{dd'}} A_\infty \times \prod_{d'' \leftarrow d'} A_{d''}\}_{d \to d'}$, where $p_{dd'} = A_\infty \times proj_{dd'}$ and $proj_{dd'}$ makes the diagram

$$\begin{tikzcd}
\prod_{d'' \leftarrow d} A_{d''} \arrow[r, "proj_{dd'}"] \arrow[d, "\pi_{d''}"'] & \prod_{d'' \leftarrow d'} A_{d''} \arrow[d, "\pi_{d''}"] \\
A_{d''} \arrow[r, "1_{A_{d''}}"'] & A_{d''}
\end{tikzcd}$$

commute. If $(f_\infty, \langle f_d \rangle) : (A_\infty, \langle A_d \rangle) \to (B_\infty, \langle B_d \rangle)$ then $R(f_\infty, \langle f_d \rangle) = \{f_\infty \times \prod_{d' \leftarrow d} f_d\}$. It is easy to show that R is right adjoint to L. Suppose $\{f_d\}, \{g_d\} : \{A_d \to A_{d'}\}_{d \to d'} \to$

$\{B_d \to B_{d'}\}_{d\to d'}$ is a parallel pair in $\boldsymbol{A}^{\boldsymbol{D}}$ such that $L(\{f_d\}), L(\{g_d\})$ has an absolute equalizer

$$(E_\infty, \langle E_d\rangle) \xrightarrow{(e_\infty, \langle e_d\rangle)} (\varinjlim_d A_d, \langle A_d\rangle) \underset{(\varinjlim_d g_d, \langle g_d\rangle)}{\overset{(\varinjlim_d f_d, \langle f_d\rangle)}{\rightrightarrows}} (\varinjlim_d B_d, \langle B_d\rangle).$$

Projecting from $\boldsymbol{A}^{|X_{\boldsymbol{D}}|}$, we obtain, for every $d \in \boldsymbol{D}$, an absolute equalizer

$$E_d \xrightarrow{e_d} A_d \underset{g_d}{\overset{f_d}{\rightrightarrows}} B_d$$

and another absolute equalizer

$$E_\infty \xrightarrow{e_\infty} \varinjlim_d A_d \underset{\varinjlim_d g_d}{\overset{\varinjlim_d f_d}{\rightrightarrows}} \varinjlim_d B_d.$$

Therefore, for every $d \to d'$ in $\boldsymbol{D}$ we can induce an arrow $E_d \to E_{d'}$ such that

$$\begin{tikzcd} E_d \arrow[r, "e_d"] \arrow[d] & A_d \arrow[d] \\ E_{d'} \arrow[r, "e_{d'}"'] & A_{d'} \end{tikzcd}$$

commutes. It is easily seen that we obtain an equalizer diagram

$$\{E_d \to E_{d'}\}_{d\to d'} \xrightarrow{\{e_d\}} \{A_d \to A_{d'}\}_{d\to d'} \underset{\{g_d\}}{\overset{\{f_d\}}{\rightrightarrows}} \{B_d \to B_{d'}\}_{d\to d'}.$$

Since filtered colimits commute with pointwise absolute equalizers we obtain that L preserves these equalizers. It is clear that L reflects these equalizers. Therefore L is cotripleable. □

If we look at the cotriple generated by the adjunction $L \dashv R$ of the lemma we obtain $\boldsymbol{G}^{X^+_{\boldsymbol{D}}}$, which means that the categories $\boldsymbol{A}^{\boldsymbol{D}}$ and $\mathcal{A}^{X^+_{\boldsymbol{D}}}$ are equivalent. Now, the comparison functor $\Phi_{\boldsymbol{D}} : \boldsymbol{A}^{\boldsymbol{D}} \to \mathcal{A}^{X^+_{\boldsymbol{D}}}$ is such that $\Phi_{\boldsymbol{D}}(\{A_d \xrightarrow{\sigma_{dd'}} A_{d'}\}_{d\to d'}) = (\varinjlim_d A_d \xrightarrow{1 \times \varinjlim_d (i_d \times \langle \sigma_{dd'}\rangle)} \varinjlim_d A_d \times \varinjlim_d (\varinjlim_d A_d \times \prod_{d''\leftarrow d} A''_d), \langle A_d \xrightarrow{\sigma_{dd'}} \prod_{d''\leftarrow d} A_{d''}\rangle)$, and $\Phi_{\boldsymbol{D}}(\{f_d\}) = (\varinjlim_d f_d, \langle f_d\rangle)$. The quasi inverse $\Psi_{\boldsymbol{D}} : \mathcal{A}^{X^+_{\boldsymbol{D}}} \to \boldsymbol{A}^{\boldsymbol{D}}$ is a lot simpler, $\Psi_{\boldsymbol{D}}(A_\infty \to \varinjlim_{U \ni \infty} \prod_{d\in U} A_d, \langle A_d \to A_\infty \times \prod_{d\to d'} A_{d'}\rangle) = \{A_d \to A_\infty \times \prod_{d'\leftarrow d} A_{d'} \xrightarrow{\pi_{d'}} A_{d'}\}_{d'\leftarrow d}$.

Corollary 6.2. *With the same hypotheses and notation as in lemma* 6.1, *the diagram*

$$\begin{tikzcd} \mathcal{A}^{X_D^+} \arrow[rr, "\Psi_D"] \arrow[dr, "\infty^*"'] & & \mathcal{A}^D \arrow[dl, "\varinjlim"] \\ & \mathcal{A} & \end{tikzcd}$$

commutes, where Ψ_D is the functor just defined.

It is easily seen that the functor $K : \mathbf{A}^D \to \mathbf{A}^{|X_D|}$ such that $K(\{A_d \xrightarrow{\sigma_{dd'}} A_{d'}\}_{d \to d'}) = \langle A_d \rangle_D$ is also cotripleable and defines the cotriple $\mathbf{G}^{X_D}$. Thus, in view of the previous corollary we have that the categories $\mathcal{A}^{X_D}$ and $\mathcal{A}^{X_D^+}$ are equivalent. In the particular case when $\mathbf{D} = 2$ we have that in X_2 1 and ∞ can not be distinguished from each other so and we will feel free to replace $\mathcal{A}^{X_2^+}$ by $\mathcal{A}^S$.

6.6 The Functor $(\_)^1$:Top-ind$(\mathcal{A}, \mathcal{B}) \to$ *Filt*$(\mathbf{A}, \mathbf{B})$

From now on we are going to suppose that $\mathbf{A}$ and $\mathbf{B}$ are categories with products and filtered colimts such that filtered colimits commute with pointwise absolute equalizers and that $\mathcal{A}$ and $\mathcal{B}$ are the ***Top***-indexed categories of coalgebras over $\mathbf{A}$ and $\mathbf{B}$ respectively.

Lemma 6.3. *If $G : \mathcal{A} \to \mathcal{B}$ is a **Top**-indexed functor then there exists a strict **Top**-indexed functor $F : \mathcal{A} \to \mathcal{B}$ isomorphic to G (in **Top**-ind$(\mathcal{A}, \mathcal{B})$).*

Proof. Let $G : \mathcal{A} \to \mathcal{B}$ be a ***Top***-indexed functor. For any X in ***Top*** and any $x \in X$, we have a continuous function $x : 1 \to X$, and a natural isomorphism $x^*G^X \to Gx^*$. Therefore, given $\langle A_x \xrightarrow{\tau_x} \varinjlim_{U \ni x} \prod_{y \in U} A_y \rangle$ in $\mathcal{A}^X$, we have a natural isomorphism $x^*G^X(\langle \tau_x \rangle) \xrightarrow{\cong} GA_x$. Define $F^X : \mathcal{A}^X \to \mathcal{B}^X$ such that $F^X(\langle \tau_x \rangle)$ is $\langle GA_x \xrightarrow{\cong} x^*G(\langle \tau_x \rangle) \to \varinjlim_{U \ni x} \prod_{y \in U} y^*G(\langle \tau_x \rangle) \xrightarrow{\cong} \varinjlim_{U \ni x} \prod_{y \in U} GA_y \rangle$. It is not hard to show that we obtain a coalgebra in this way and that the functor F is strict and isomorphic to G. $\square$

In view of this theorem we will assume that our ***Top***-indexed functors are strict.

Lemma 6.4. *If $F : \mathcal{A} \to \mathcal{B}$ is a* **Top***-indexed functor, then the square*

$$\begin{tikzcd}
\mathcal{A}^S \arrow[r, "\Psi_2"] \arrow[d, "F^S"'] & \mathcal{A}^2 \arrow[d, "F^2"] \\
\mathcal{B}^S \arrow[r, "\Psi_2"'] & \mathcal{B}^2
\end{tikzcd}$$

commutes up to isomorphism.

Proof. We are using $\mathcal{A}^S$ instead of $\mathcal{A}^{X_2^+}$. Now, consider the continuous maps

$$\begin{tikzcd}
1 \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "0"'] & S \arrow[l, "t"']
\end{tikzcd}$$

S. These maps induce the diagram

$$\begin{tikzcd}
\mathcal{B}^S \arrow[rrr, "\Psi_2"] \arrow[ddd, shift right=3, "0^*"'] \arrow[ddd, shift left=3, "1^*"] & & & \mathcal{B}^2 \arrow[ddd, shift right=3, "\delta_0"'] \arrow[ddd, shift left=3, "\delta_1"] \\
& \mathcal{A}^S \arrow[ul, "F^S"'] \arrow[r, "\Psi_2"] \arrow[d, shift right=3, "0^*"'] \arrow[d, shift left=3, "1^*"] & \mathcal{A}^2 \arrow[ur, "F^2"] \arrow[d, shift right=3, "\delta_0"'] \arrow[d, shift left=3, "\delta_1"] & \\
& A \arrow[r, "1_A"] \arrow[u, "t^*"] \arrow[dl, "F"'] & A \arrow[u, "id"] \arrow[dr, "F"] & \\
B \arrow[rrr, "1_B"'] \arrow[uuu, "t^*"] & & & B \arrow[uuu, "id"]
\end{tikzcd}$$

in which it is easy to see that the front and back faces commute sequentially, and the sides commute as well. Then it is not hard to see that the top commutes as well. □

Lemma 6.5. *For any directed poset $\mathbf{D}$, the following diagram commutes*

$$\begin{tikzcd}
\mathcal{A}^{X_D^+} \arrow[r, "\Psi_D"] \arrow[d, "F^{X_D^+}"'] & A^D \arrow[d, "F^D"] \\
\mathcal{B}^{X_D^+} \arrow[r, "\Psi_D"'] & B^D
\end{tikzcd}$$

Proof. Let $d \to d'$ be an arrow in $\boldsymbol{D}$, consider the functor $\alpha_{dd'} : 2 \to \boldsymbol{D}$ such that $(0 \to 1) \mapsto (d \to d')$. Consider the continuous function $\beta_{dd'} : S \to X_{\boldsymbol{D}}$, such that $\beta(0) = d$ and $\beta(1) = d'$. Then it is easy to see that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{A}^S \arrow[rrr, "\Psi_2"] \arrow[ddd, "F^S"'] & & & \boldsymbol{A}^2 \arrow[ddd, "F^2"] \\
& \mathcal{A}^{X_{\boldsymbol{D}}^+} \arrow[ul, "\beta^*_{dd'}"'] \arrow[r, "\Psi_{\boldsymbol{D}}"] \arrow[d, "F^{X_{\boldsymbol{D}}^+}"'] & \boldsymbol{A}^{\boldsymbol{D}} \arrow[ur, "\boldsymbol{A}^{\alpha_{dd'}}"] \arrow[d, "F^{\boldsymbol{D}}"] & \\
& \mathcal{B}^{X_{\boldsymbol{D}}^+} \arrow[dl, "\beta^*_{dd'}"'] \arrow[r, "\Psi_{\boldsymbol{D}}"'] & \boldsymbol{B}^{\boldsymbol{D}} \arrow[dr, "\boldsymbol{B}^{\alpha_{dd'}}"] & \\
\mathcal{B}^S \arrow[rrr, "\Psi_2"'] & & & \boldsymbol{B}^2
\end{tikzcd}$$

□

The following proposition is an immediate corollary of these lemmas.

Proposition 6.6. *If $F : \mathcal{A} \to \mathcal{B}$ is a* **Top***-indexed functor, then the functor $F^1 : \boldsymbol{A} \to \boldsymbol{B}$ preserves filtered colimits.*

Proof. It is enough, see [1], to show that F^1 preserves directed colimits. Consider the diagram

$$\begin{tikzcd}
\mathcal{B}^{X_{\boldsymbol{D}}^+} \arrow[rrrr, "\Psi_{\boldsymbol{D}}"] \arrow[dddrr, "\infty^*"'] & & & & \boldsymbol{B}^{\boldsymbol{D}} \arrow[dddll, "\underrightarrow{lim}"] \\
& \mathcal{A}^{X_{\boldsymbol{D}}^+} \arrow[ul, "F^{X_{\boldsymbol{D}}^+}"] \arrow[rr, "\Psi_{\boldsymbol{D}}"] \arrow[dr, "\infty^*"] & & \boldsymbol{A}^{\boldsymbol{D}} \arrow[ur, "F^{\boldsymbol{D}}"] \arrow[dl, "\underrightarrow{lim}"'] & \\
& & \boldsymbol{A} \arrow[d, "F^1"] & & \\
& & \boldsymbol{B} & &
\end{tikzcd}$$

□

The proposition allows us to define a functor $(\ )^1 : \boldsymbol{Top}\text{-}ind(\mathcal{A}, \mathcal{B}) \to \boldsymbol{Filt}(\boldsymbol{A}, \boldsymbol{B})$ such that $F \mapsto F^1$ and $\tau \mapsto \tau^1$ for every

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{B}
\end{tikzcd}$$

in $\boldsymbol{Top}\text{-ind}(\mathcal{A},\mathcal{B})$.

6.7 The Functor $\widehat{(\_)} : \boldsymbol{Filt}(\boldsymbol{A},\boldsymbol{B}) \to \textbf{Top-ind}(\mathcal{A},\mathcal{B})$

We define now a functor in the other direction. Given $H \in \boldsymbol{Filt}(\boldsymbol{A},\boldsymbol{B})$ and a topological space X, we define $\widehat{H}^X : \mathcal{A}^X \to \mathcal{B}^X$ such that

$$\widehat{H}^X(\langle A_x \xrightarrow{\alpha_x} \varinjlim_{U \ni x} \prod_{y \in U} A_y \rangle) =$$

$$\langle HA_x \xrightarrow{H\alpha_x} H(\varinjlim_{U \ni x} \prod_{y \in U} A_y) \xrightarrow{\cong} \varinjlim_{U \ni x} H(\prod_{y \in U} A_y) \to \varinjlim_{U \ni x} \prod_{y \in U} HA_y \rangle,$$

where the last arrow is the unique one that makes

$$\begin{tikzcd}
 & H(\prod_{y \in U} A_y) \arrow[r, "i_U"] \arrow[dl, "H\pi_y"'] \arrow[dd] & \varinjlim_{U \ni x} H(\prod_{y \in U} A_y) \arrow[dd] \\
HA_y & & \\
 & \prod_{y \in U} HA_y \arrow[r, "i_U"'] \arrow[ul, "\pi_y"] & \varinjlim_{U \ni x} \prod_{y \in U} HA_y
\end{tikzcd}$$

commute, and $\widehat{H}(\langle f_x \rangle_x) = \langle Hf_x \rangle_x$. It is not hard to show that we obtain coalgebras and coalgebra morphisms with the above definitions. $\widehat{H}$ turns out to be a strict $\boldsymbol{Top}$-indexed functor. We will show that, with the proper conditions on $\boldsymbol{A}$ and $\boldsymbol{B}$, the functors defined above give an equivalence of categories. Before the proof we need some lemmas.

6.8 The Ultraproduct Transition Morphisms

Suppose $F : \mathcal{A} \to \mathcal{B}$ is a $\boldsymbol{Top}$-indexed functor. Let X be topological space. For every $x \in X$ we have the continuous function $x : 1 \to X$ that sends the only element of 1 to x. This function induces the following commutative diagram

$$\begin{tikzcd}
\mathcal{A}^X \arrow[r, "x^*"] \arrow[d, "F^X"'] & \boldsymbol{A} \arrow[d, "F^1"] \\
\mathcal{B}^X \arrow[r, "x^*"'] & \boldsymbol{B}.
\end{tikzcd}$$

If we start with a coalgebra $\langle A_x \xrightarrow{\tau_x} \varinjlim_{U \ni x} \prod_{y \in U} A_y \rangle$ in $\mathcal{A}^X$ we have that

$$x^*(F^X(\langle A_x \xrightarrow{\tau_x} \varinjlim_{U \ni x} \prod_{y \in U} A_y \rangle)) = F^1(A_x).$$

This tells us that $F^X(\langle A_x \xrightarrow{\tau_x} \varinjlim_{U \ni x} \prod_{y \in U} A_y \rangle)$ is of the form

$$\langle F^1 A_x \to \varinjlim_{U \ni x} \prod_{y \in U} F^1 A_y \rangle.$$

In particular, when we have an ultrafilter $(I, \mathcal{G})$ and and a family $\langle A_i \rangle_I$ in $\mathbf{A}^I$, we obtain the coalgebra

$$\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j \xrightarrow{1} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j,$$

in $\mathcal{A}^{I_\mathcal{G}}$. Then

$$F^{I_\mathcal{G}}(\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_i \xrightarrow{1} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_i) : F^1(\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_i) \to \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} F^1 A_i.$$

We call this morphism $\gamma_{F\mathcal{G}} \langle A_i \rangle_I$. It is not hard to see that $\gamma_{F\mathcal{G}}$ defines a natural transformation

$$\begin{tikzcd}
\mathbf{A}^I \arrow[r, "\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} (\text{-})"] \arrow[d, "(F^1)^I"'] & \mathbf{A} \arrow[d, "F^1"] \arrow[dl, "\gamma_{F\mathcal{G}}"'] \\
\mathbf{B}^I \arrow[r, "\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} (\text{-})"'] & \mathbf{B}.
\end{tikzcd}$$

Lemma 6.7. *If $F : \mathcal{A} \to \mathcal{B}$ is a **Top**-indexed functor then for every ultrafilter $(I, \mathcal{G})$ we have that*

$$F^{I_\mathcal{G}}(A_{a_\mathcal{G}} \xrightarrow{\sigma} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j) = F(A_{a_\mathcal{G}}) \xrightarrow{F(\sigma)} F(\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j) \xrightarrow{\gamma_{F\mathcal{G}}} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} F A_j.$$

Proof. Given $A_{a_\mathcal{G}} \xrightarrow{\sigma} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j$, consider the morphism

$$\begin{tikzcd}
A_{a_\mathcal{G}} \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j \arrow[d, "\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} 1_{A_j}"] \\
\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j \arrow[r, "1"'] & \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j
\end{tikzcd}$$

in $\mathcal{A}^{I_\mathcal{G}}$ and apply $F^{I_\mathcal{G}}$. $\square$

6.9 Reduced Products and Ultraproducts

Finally, we need a condition on $\boldsymbol{B}$. Given a filter $(I, \mathcal{F})$, define $\mathcal{U}_{\mathcal{F}} = \{\mathcal{G} | \mathcal{G}$ is an ultrafilter on I and $\mathcal{F} \subset \mathcal{G}\}$.

Definition 6.4. We say that ultraproducts determine reduced products in $\boldsymbol{B}$ if for every filter $(I, \mathcal{F})$ and every $\langle B_i \rangle_I \in \boldsymbol{B}^I$ we have that the family $\{\varinjlim_{J \in \mathcal{F}} \prod_{j \in J} B_j \xrightarrow{i_{\mathcal{F}\mathcal{G}}} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} B_j\}_{\mathcal{G} \in \mathcal{U}_{\mathcal{F}}}$ is jointly monic, where $i_{\mathcal{F}\mathcal{G}}$ makes the diagram

$$\begin{tikzcd} \prod_J B_j \arrow[r, "i_J"] \arrow[dr, "i_J"'] & \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} B_j \arrow[d, "i_{\mathcal{F}\mathcal{G}}"] \\ & \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} B_j \end{tikzcd}$$

commute for every $J \in \mathcal{F}$.

Using the fact that for every filter $(I, \mathcal{F})$ we have that $\mathcal{F} = \bigcap_{\mathcal{G} \in \mathcal{U}_{\mathcal{F}}} \mathcal{G}$, it is not hard to prove that the condition above is true for the category ***Set***.

Lemma 6.8. *If in $\boldsymbol{B}$ reduced products are determined by ultraproducts and $F : \mathcal{A} \to \mathcal{B}$ is a **Top**-indexed functor, then F is determined by the natural transformations $\gamma_{F\mathcal{G}}$ for all ultrafilters $(I, \mathcal{G})$.*

Proof. Let $(I, \mathcal{F})$ be a filter, and $\mathcal{G} \in \mathcal{U}_{\mathcal{F}}$. Now consider $F : \mathcal{A} \to \mathcal{B}$, and the continuous function $h_{\mathcal{G}\mathcal{F}} : I_{\mathcal{G}} \to I_{\mathcal{F}}$ defined after definition 6.3. We have then that the following diagram

$$\begin{tikzcd} \mathcal{A}^{I_{\mathcal{F}}} \arrow[r, "h^*_{\mathcal{G}\mathcal{F}}"] \arrow[d, "F^{I_{\mathcal{F}}}"'] & \mathcal{A}^{I_{\mathcal{G}}} \arrow[d, "F^{I_{\mathcal{G}}}"] \\ \mathcal{B}^{I_{\mathcal{F}}} \arrow[r, "h^*_{\mathcal{G}\mathcal{F}}"'] & \mathcal{B}^{I_{\mathcal{G}}} \end{tikzcd}$$

commutes. Following the image of an arbitrary $A_{a_{\mathcal{F}}} \xrightarrow{\sigma} \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} A_j$ we have that $F^{I_{\mathcal{G}}}(A_{a_{\mathcal{F}}} \xrightarrow{\sigma} \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} A_j \xrightarrow{i_{\mathcal{F}\mathcal{G}}} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j)$ is equal to the composition $FA_{a_{\mathcal{F}}} \xrightarrow{F^{I_{\mathcal{F}}}(\sigma)}$

$\varinjlim_{J \in \mathcal{F}} \prod_{j \in J} FA_j \xrightarrow{i_{\mathcal{F}\mathcal{G}}} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} FA_j$. Or put another way, we have that

$$\begin{tikzcd} FA_{a_{\mathcal{F}}} \arrow[r, "F(i_{\mathcal{F}\mathcal{G}} \circ \sigma)"] \arrow[d, "F^{I_{\mathcal{F}}}(\sigma)"'] & F(\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j) \arrow[d, "\gamma_{\mathcal{F}\mathcal{G}}"] \\ \varinjlim_{J \in \mathcal{F}} \prod_{j \in J} FA_j \arrow[r, "i_{\mathcal{F}\mathcal{G}}"'] & \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} FA_j \end{tikzcd}$$

commutes. Since the family $\{i_{\mathcal{F}\mathcal{G}}\}_{\mathcal{G} \in \mathcal{U}_{\mathcal{F}}}$ is jointly monic, we have that $F^{I_{\mathcal{F}}}(\sigma)$ is determined by the natural transformations $\gamma_{\mathcal{F}\mathcal{G}}$ with $\mathcal{G} \in \mathcal{U}_{\mathcal{F}}$. Now, given a topological space X, and a point $x \in X$, let $I = X - \{x\}$ and $\mathcal{F}_x = \{J \subset I | J \cup x$ is a neighbourhood of $x\}$. $\mathcal{F}_x$ is a filter on I and there is a continuous function $h : I_{\mathcal{F}_x} \to X$ such that $h|_I$ is the inclusion and $h(a_{\mathcal{F}_x}) = x$. Then we have a commutative square

$$\begin{tikzcd} \mathcal{A}^X \arrow[r, "h^*"] \arrow[d, "F^X"'] & \mathcal{A}^{I_{\mathcal{F}_x}} \arrow[d, "F^{I_{\mathcal{F}_x}}"] \\ \mathcal{B}^X \arrow[r, "h^*"'] & \mathcal{B}^{I_{\mathcal{F}_x}} \end{tikzcd}$$

Following the image of an arbitrary coalgebra we see that F^X is determined by $\{F^{I_{\mathcal{F}_x}}\}_{x \in X}$. □

6.10 Top-ind$(\mathcal{A}, \mathcal{B})$ equivalent to $\boldsymbol{Filt}(\boldsymbol{A}, \boldsymbol{B})$

Proposition 6.9. *Let $\mathbf{A}$ and $\mathbf{B}$ be categories with products and filtered colimits such that directed colimits commute with pointwise absolute equalizers, and such that reduced products are determined by ultraproducts in $\mathbf{B}$, then the category $\boldsymbol{Top}$-ind$(\mathcal{A}, \mathcal{B})$ is equivalent to the category $\boldsymbol{Filt}(\mathbf{A}, \mathbf{B})$ of functors from $\mathbf{A}$ to $\mathbf{B}$ that preserve filtered colimits.*

Proof. We have already defined the functors $(\ )^1 : \boldsymbol{Top}\text{-ind}(\mathcal{A}, \mathcal{B}) \to \boldsymbol{Filt}(\mathbf{A}, \mathbf{B})$ and $\widehat{(\ )} : \boldsymbol{Filt}(\mathbf{A}, \mathbf{B}) \to \boldsymbol{Top}\text{-ind}(\mathcal{A}, \mathcal{B})$. It is clear that $(\ )^1 \circ \widehat{(\ )}$ is the identity. Let $F : \mathcal{A} \to \mathcal{B}$ be a $\boldsymbol{Top}$-indexed functor, we will show that for every ultrafilter $(I, \mathcal{G})$ and every $\langle A_i \rangle_I, \gamma_{\mathcal{F}\mathcal{G}}(\langle A_i \rangle_I)$ is

$$F^1(\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j) \xrightarrow{\cong} \varinjlim_{J \in \mathcal{G}} F^1(\prod_{j \in J} A_j) \to \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} F^1 A_j.$$

Thus, using lemma 6.8 we conclude that $F = \widehat{F^1}$.

Let $(I, \mathcal{G})$ be an ultrafilter, and $J_0 \in \mathcal{G}$. Then $\mathcal{S}(J_0)$ denotes the principal filter on I generated by J_0. For every $j \in J_0$ we have the continuous function $h_{jJ_0} : S \to I_{\mathcal{S}(J_0)}$ defined after definition 6.3, that induces the following commutative square

$$\begin{tikzcd}
\mathcal{A}^{I_{\mathcal{S}(J_0)}} \arrow[r, "h^*_{jJ_0}"] \arrow[d, "F^{I_{\mathcal{S}(J_0)}}"'] & \mathcal{A}^S \arrow[d, "F^S"] \\
\mathcal{B}^{I_{\mathcal{S}(J_0)}} \arrow[r, "h^*_{jJ_0}"'] & \mathcal{B}^S.
\end{tikzcd}$$

If we start with $(\langle A_i \rangle, A_{a_{\mathcal{S}(J_0)}} \xrightarrow{\langle m_j \rangle} \prod_{j' \in J_0} A_{j'}) \in \mathcal{A}^{I_{\mathcal{S}(J_0)}}$, then we have that $F(m_j) = (F^{I_{\mathcal{S}(J_0)}}(\langle A_i \rangle, A_{a_{\mathcal{S}(J_0)}} \xrightarrow{\langle m_j \rangle} \prod_{j' \in J_0} A_{j'}))_j$. Therefore

$$F^{I_{\mathcal{S}(J_0)}}(\langle A_i \rangle, A_{a_{\mathcal{S}(J_0)}} \xrightarrow{\langle m_j \rangle} \prod_{j' \in J_0} A_{j'}) = (\langle F A_i \rangle, F A_{a_{\mathcal{S}(J_0)}} \xrightarrow{\langle F m_j \rangle} \prod_{j' \in J_0} F A_{j'}).$$

Now, the continuous function $h_{\mathcal{G}\mathcal{S}(J_0)} : I_{\mathcal{G}} \to I_{\mathcal{S}(J_0)}$ induces another commutative square

$$\begin{tikzcd}
\mathcal{A}^{I_{\mathcal{S}(J_0)}} \arrow[r, "h^*_{J_0}"] \arrow[d, "F^{I_{\mathcal{S}(J_0)}}"'] & \mathcal{A}^{I_{\mathcal{G}}} \arrow[d, "F^{I_{\mathcal{G}}}"] \\
\mathcal{B}^{I_{\mathcal{S}(J_0)}} \arrow[r, "h^*_{J_0}"'] & \mathcal{B}^{I_{\mathcal{G}}}
\end{tikzcd}$$

from which we conclude that

$$F^{I_{\mathcal{G}}}(A_{a_{\mathcal{S}(J_0)}} \xrightarrow{\langle m_j \rangle} \prod_{j \in J_0} A_j \xrightarrow{i_{J_0}} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j) = F A_{a_{\mathcal{S}\mathcal{S}(J_0)}} \xrightarrow{\langle F m_j \rangle} \prod_{j \in J_0} F A_j \xrightarrow{i_{J_0}} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} F A_j.$$

In particular, taking $A_{a_{\mathcal{S}(J_0)}} = \prod_{j \in J_0} A_j$ and $m_j = \pi_j$, consider the morphism

$$\begin{tikzcd}
\prod_{j \in J_0} A_j \arrow[r, "i_{J_0}"] \arrow[d, "i_{J_0}"'] & \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j \arrow[d, "1"] \\
\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j \arrow[r, "1"'] & \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j
\end{tikzcd}$$

in $\mathcal{A}^{I_{\mathcal{G}}}$, apply $F^{I_{\mathcal{G}}}$ to obtain that

$$F^{I_{\mathcal{G}}}(\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j \xrightarrow{1} \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j) =$$

$$F(\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} A_j) \xrightarrow{\cong} \varinjlim_{J \in \mathcal{G}} F(\prod_{j \in J} A_j) \to \varinjlim_{J \in \mathcal{G}} \prod_{j \in J} FA_j.$$

This last arrow is then $\gamma_{F\mathcal{G}}$. Since we already know that F is determined by these arrows we see that we have an equivalence as stated. □

6.11 Subcategories Closed Under Ultraproducts

Suppose now that we have a full subcategory $\boldsymbol{A}_0$ of $\boldsymbol{A}$ such that $\boldsymbol{A}_0$ has filtered colimits and they are preserved by the inclusion $\boldsymbol{A}_0 \to \boldsymbol{A}$. Then we can define a sub $\boldsymbol{Top}$-indexed category $\mathcal{A}_0$ of $\mathcal{A}$ as follows. $\mathcal{A}_0^X$ is the full subcategory of $\mathcal{A}^X$ whose objects are the coalgebras $\langle A_x \xrightarrow{\tau_x} \varinjlim_{U \ni x} \prod_{y \in U} A_y \rangle$ such that for every $x \in X$ we have that A_x is an object of $\boldsymbol{A}_0$. It is clear that for every continuous function $f : Z \to X$, the functor $f^* : \mathcal{A}^X \to \mathcal{A}^Z$ restricts to $\mathcal{A}_0^X$, that is, $f^* : \mathcal{A}_0^X \to \mathcal{A}_0^Z$. It also is clear that for every directed poset $\boldsymbol{D}$, the functor $\Psi_{\boldsymbol{D}} : \mathcal{A}^{X_{\boldsymbol{D}}^+} \to \boldsymbol{A}^{\boldsymbol{D}}$ restricts to $\Psi_{\boldsymbol{D}} : \mathcal{A}_0^{X_{\boldsymbol{D}}^+} \to \mathcal{A}_0^{\boldsymbol{D}}$.

We will be able to apply the results of this section to $\boldsymbol{Top}$-indexed categories of models due to the fact that models over a sheaf category are the same thing as sheaves of models as the next proposition shows

Proposition 6.10. *The category of models $\mathcal{MOD}(\boldsymbol{P})^X$ is equivalent to the full subcategory of $(\mathcal{SET}^{\boldsymbol{P}})^X$ whose objects are coalgebras $\langle M_x \xrightarrow{\tau_x} \varinjlim_{U \ni x} \prod_{y \in U} M_y \rangle$ such that for every $x \in X$, $M_x \in \boldsymbol{Mod}(\boldsymbol{P})$.*

Proof. First notice that this is clearly true for the topological space 1. Given a topological space X, a model $M \in \mathcal{MOD}(\boldsymbol{P})^X$ corresponds to the coalgebra $\langle x^*M \to \varinjlim_{U \ni x} \prod_{y \in U} y^*M \rangle$ in $(\mathcal{SET}^{\boldsymbol{P}})^X$. Clearly $x^*M \in \boldsymbol{Mod}(\boldsymbol{P})$. On the other hand, if we start with a coalgebra $\langle M_x \xrightarrow{\tau_x} \varinjlim_{U \ni x} \prod_{y \in U} M_y \rangle$ in $(\mathcal{SET}^{\boldsymbol{P}})^X$ such that for every $x \in X$ we have that $M_x \in \boldsymbol{Mod}(\boldsymbol{P})$, this determines a functor $M : \boldsymbol{P} \to Sh(X)$ such that $MP = \langle M_x P \xrightarrow{\tau_x P} \varinjlim_{U \ni x} \prod_{y \in U} M_y P \rangle$. □

Definition 6.5. We say that the subcategory $\boldsymbol{A}_0$ is closed under $\boldsymbol{A}$-ultraproducts if for every ultrafilter $(I, \mathcal{G})$ we have that the functor $\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} (\_) : \boldsymbol{A}^I \to \boldsymbol{A}$ restricts to a functor $\varinjlim_{J \in \mathcal{G}} \prod_{j \in J} (\_) : \boldsymbol{A}_0^I \to \boldsymbol{A}_0$.

Fix full subcategories $\boldsymbol{A}_0$ of $\boldsymbol{A}$, and $\boldsymbol{B}_0$ of $\boldsymbol{B}$, with filtered colimits preserved by both inclusions and such that $\boldsymbol{A}_0$ is closed under $\boldsymbol{A}$-ultraproducts and $\boldsymbol{B}_0$ is closed under $\boldsymbol{B}$-ultraproducts. Define $\mathcal{A}_0$ and $\mathcal{B}_0$ as above. We assume as well that in $\boldsymbol{A}$ and in $\boldsymbol{B}$ filtered colimits commute with pointwise absolute equalizers.

Lemma 6.11. *If $F : \mathcal{A}_0 \to \mathcal{B}_0$ is a **Top**-indexed functor, then $F^1 : \boldsymbol{A}_0 \to \boldsymbol{B}_0$ preserves filtered colimits.*

Proof. We can repeat the same reasoning that leads to the proof of proposition 6.6. □

We have then a functor $(\ )^1 : \boldsymbol{Top}\text{-}ind(\mathcal{A}_0, \mathcal{B}_0) \to \boldsymbol{Filt}(\boldsymbol{A}_0, \boldsymbol{B}_0)$. Notice that we can not define a functor in the other direction as before because we do not have, in general, products in $\boldsymbol{A}_0$ or $\boldsymbol{B}_0$.

Given $F : \mathcal{A}_0 \to \mathcal{B}_0$, we can define the natural transformations $\gamma_{F\mathcal{G}}$ for every ultrafilter $(I, \mathcal{G})$ as before, that is, $\gamma_{F\mathcal{G}}\langle A_i \rangle_I$ is

$$F^{I\mathcal{G}}\big(\varinjlim_{J\in\mathcal{G}} \prod_{j\in J} A_i \xrightarrow{1} \varinjlim_{J\in\mathcal{G}} \prod_{j\in J} A_i\big) : F^1\big(\varinjlim_{J\in\mathcal{G}} \prod_{j\in J} A_i\big) \to \varinjlim_{J\in\mathcal{G}} \prod_{j\in J} F^1 A_i.$$

or put in a diagram

$$\begin{tikzcd}
\boldsymbol{A}_0^I \arrow[rr, "\varinjlim_{J\in\mathcal{G}} \prod_{j\in J}(\text{-})"] \arrow[dd, "(F^1)^I"'] & & \boldsymbol{A}_0 \arrow[dd, "F^1"] \arrow[ddl, phantom, "\swarrow \gamma_{F\mathcal{G}}"] \\
 & & \\
\boldsymbol{B}_0^I \arrow[rr, "\varinjlim_{J\in\mathcal{G}} \prod_{j\in J}(\text{-})"'] & & \boldsymbol{B}_0.
\end{tikzcd}$$

With essentially the same proof we also have

Lemma 6.12. *If in $\boldsymbol{B}$ reduced products are determined by ultraproducts and $F : \mathcal{A}_0 \to \mathcal{B}_0$ is a **Top**-indexed functor, then F is determined by the natural transformations $\gamma_{F\mathcal{G}}$ for all ultrafilters $(I, \mathcal{G})$.*

□

Lemma 6.13. *The functor $(\ )^1 : \boldsymbol{Top}\text{-}ind(\mathcal{A}_0, \mathcal{B}_0) \to \boldsymbol{Filt}(\boldsymbol{A}_0, \boldsymbol{B}_0)$ is faithful.*

Proof. If $\phi : F \to G : \mathcal{A}_0 \to \mathcal{B}_0$ is a $\boldsymbol{T}$-indexed natural transformation, X is a topological space and $z \in X$, consider the following diagram

$$\begin{tikzcd}
\mathcal{A}_0^X \arrow[r, bend left, "F^X"] \arrow[r, bend right, "G^X"'] \arrow[r, phantom, "\Downarrow\phi^X"] \arrow[d, "z^*"'] & \mathcal{B}_0^X \arrow[d, "z^*"] \\
\mathcal{A}_0 \arrow[r, bend left, "F^1"] \arrow[r, bend right, "G^1"'] \arrow[r, phantom, "\Downarrow\phi^1"] & \mathcal{B}_0.
\end{tikzcd}$$

Since ϕ is a $\boldsymbol{T}$-indexed natural transformation, we have that for any coalgebra $\langle A_x \xrightarrow{\tau_x} \underrightarrow{lim}_{U \ni x} \prod_{y \in U} A_y \rangle$ in $\mathcal{A}_0^X$, $(\phi^X \langle \tau_x \rangle)_z = \phi^1 A_z : FA_z \to GA_z$. It is clear then that ϕ^X is totally determined by ϕ^1 □

It is easy to see that for every small pretopos $\boldsymbol{P}$ the category $\boldsymbol{Mod}(\boldsymbol{P})$ satisfies all the necesary conditions as a full subcategory of $\boldsymbol{Set}^{\boldsymbol{P}}$ and therefore as a corollary of lemma 6.11 we have

Proposition 6.14. *For any* **Top**-*indexed functor* $F : \mathcal{MOD}(\boldsymbol{P}) \to \mathcal{MOD}(\boldsymbol{Q})$ *the functor* $F^1 : \boldsymbol{Mod}(\boldsymbol{P}) \to \boldsymbol{Mod}(\boldsymbol{Q})$ *preserves filtered colimits.* □

Bibliography

[1] M. Artin, A. Grothendieck, and J.L. Verdier, *Théorie des topos et cohomologie étale des schémas*, vol. 269, Springer-Verlag, 1972.

[2] M. Barr and C. Wells, *Toposes, triples and theories*, Springer-Verlag, 1985.

[3] J. Benabou, *Fibrations petites et localment petites*, C. R. Acad. Sc. Paris **281** (1975), 897–900.

[4] ———, *Théories relatives à un corpus*, C. R. Acad. Sc. Paris **281** (1975), 831–834.

[5] R. Blackwell, G. M. Kelly, and J. Power, *Two dimensional monad theory*, Journal of Pure and Applied Algebra **59** (1989), 1–41.

[6] J. Dionne, *Des théories élémentaires aux catégories conceptuelles*, Master's thesis, Université de Montréal, 1973.

[7] J. R. Isbell, *Structure of categories*, Bulletin of the AMS **72** (1966), 619–655.

[8] P. T. Johnstone, *Topos theory*, Academic Press, 1977.

[9] G. M. Kelly and R. Street, *Review of the elements of 2-categories*, Lecture Notes in Mathematics 420 (1974), 75–103.

[10] A. Kock and G. E. Reyes, *Doctrines in categorical logic*, Handbook of Mathematical Logic, Studies in Logic and the Foundations of Mathematics, vol. 90, North-Holland, 1977.

[11] D. Lever, *Continous families, topological categories, and toposes*, Ph.D. thesis, Dalhousie University, 1981.

[12] F. E. J. Linton, *Coequalizers in categories of algebras*, Seminar on Triples and Categorical Homology Theory, Lecture Notes in Mathematics, vol. 80, Springer-Verlag, 1969.

[13] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, 1971.

[14] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic*, Springer-Verlag, 1992.

[15] M. Makkai, *Stone duality for first order logic*, Advances in Mathematics **65** (1987), no. 2, 97–170.

[16] M. Makkai and R. Paré, *Accessible categories: The foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, 1989.

[17] M. Makkai and A. M. Pitts, *Some results on locally finitely presentable categories*, Transactions of the AMS **299** (1987), no. 2, 473–495.

[18] M. Makkai and G. E. Reyes, *First order categorical logic*, Lecture Notes in Mathematics, vol. 611, Springer-Verlag, 1977.

[19] R. Paré and D. Schumacher, *Abstract families and the adjoint functor theorem*, Indexed Categories and Their Applications, Springer-Verlag, 1978.